Technical Note GKSS/WMS/00/03
interner Bericht

Finite Element Modelling

Lecture Notes
Faculty of Engineering Kiel

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Inhalte Wiederholung von Grundbegriffen der Festigkeitslehre und Einführung in die Kontinuumsmechanik fester Körper: Spannungs- und Verzerrungstensor, Gleichgewichtsbedingungen, elastisches Stoffgesetz,

Formulierung des Festigkeitsproblems als Randwertaufgabe und als Variationsaufgabe (Prinzip der virtuellen Arbeiten)

Einführung in die Methode der finiten Elemente (FEM): Diskretisierung des Kontinuums, Verschiebungsansätze in den Elementen, (elastisches) Stoffgesetz, Steifigkeitsmatrix und Lastvektoren, Lösung des Gleichungssystems

Aufbau eines FE-Programmes: erforderliche Eingaben, Programmablauf, Ergebnisdaten

Einführung in die Anwendung des FE-Programmsystems ANSYS mit Übungen am Rechner: Erstellung von FE-Netzen und sonstigen Eingabedaten, Durchführung von FE-Rechnungen, Ergebnisdarstellung und Ergebnisauswertung.

Selbständige Bearbeitung einfacher Probleme: Stab unter Zugbelastung, Biegebelastung einer eingespannten Scheibe mit Rechteckquerschnitt, gelochte Scheibe unter zweiachsialem Zug.

empfohlene Literatur (deutsch):

J. ALTENBACH und H. ALTENBACH: "Einführung in die Kontinuumsmechanik", Teubner Studienbücher Mechanik, Stuttgart 1994.

K. KNOTHE, H. WESSELS: "Finite Elemente, Einführung für Ingenieure", Springer-Verlag, Berlin, 1991.

K.J. BATHE: "Finite-Elemente-Methoden", Springer-Verlag, Berlin, 1986.

O.C. ZIENKIEWICZ: "Methode der Finiten Elemente", Carl-Hanser-Verlag, München, 1984.

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Contents repetition of fundamentals of strength of materials and introduction into continuum mechanics of solids: stress and strain tensors, balance equantions, constitutive equations of linear elasticity,

formulation as boundary value problem and variational problem (principle of virtual work)

introduction into the finite element method (FEM): discretization of the continuum, shape and displacement functions, isoparametric elements, (elastic) constitutive equations, local and global stiffnes matrices and load vectors, solution of equations

general structure of a FE programme: input data, flow diagram of problem solution, results

application of the FE programme ANSYS with computer exercises: meshes and other input data, FE simulations, presentation, evaluation and interpretation of results,

individual working on case and parameter studies: tensile bars, bending of a clamped rectangular panel, panel with hole under biaxial tension

recommended literature (english):

W.M. LAI, D. RUBIN, and E. KREMPL: "Introduction to Continuum Mechanics", Pergamon Press, 1993.

M.E. GURTIN: "An introduction to Continuum Mechanics", Academic Press, New York, 1981.

K.J. BATHE, "Finite Element Procedures", Prentice Hall, New Jersey 07632, 1996.

O.C. ZIENKIEWICZ: "The Finite Elemente Method", McGraw-Hill Book Comp., Maidenhead (UK), 1977.

Contents

Continuum Mechanics

Continuum mechanics is a

phenomenological field theory.

Basing on observed phenomena, mathematical models for the mechanical behaviour of matter are formulated.

As everybody knows, the behaviour of matter is determined by interactions of atoms and molecules. However, an engineering modeling cannot be done on this level.

The discretely structured matter is hence represented by

a **phenomenological model**, i.e. the **continuum**;

this is done by averaging its properties in space.

For describing the mechanical behaviour of heterogeneous materials with strong local gradients of the microstructure a phenomenological theory is not sufficient, in general. For this purpose,

micromechanical models on a "**meso level**"

between a microstructural and a phenomenological modeling are developed and applied.

Equations of Continuum Mechanics

1. material independent principles

kinematics

- body
- configuration
- motion
- deformation

kinetics

- external actions: forces
- internal reactions: stresses

balance equations (conservation laws)

- matter
- momentum
- moment of momentum
- energy / work / power
- entropy (2nd principle of thermodynamics)

2. material dependent equations

constitutive equations: solids, fluids, gases
reversible, irreversible processes

> The combination of all principles leads to the formulation of initial boundary value problems.

Kinematics

Body

A **body** B is a three-dimensional differentiable manifold, the elements of which are called particles X $\in$ B.

A **body** B is endowed with a non-negative scalar measure, *m*, which is called the **mass** of the body.

Configuration

A **configuration** $\boldsymbol{\chi}$ of a body B is a smooth homeomorphism of B onto a region, $\mathbb{B} \subset \mathbb{E}^3$, of three-dimensional EUKLIDean space $\mathbb{E}^3$, $\mathbf{x} = \boldsymbol{\chi}(X)$, called the region *occupied* by the body B in the configuration $\boldsymbol{\chi}$.

Motion

A **motion of** a body B is a one-parameter family ${}^t\boldsymbol{\chi}$ of configurations, $\mathbf{x} = {}^t\boldsymbol{\chi}(X) = \boldsymbol{\chi}(X,t)$. The real parameter *t* is the **time**.

Deformation

The "**local**" **deformation** results from observing an infinitesimal vicinity of a particle in its present configuration with respect to a reference configuration.

Deformation

Any general motion of a body includes a deformation which can be described as follows. A particle, X, which is identified by its place ${}^0\mathbf{x} = {}^0\chi(\mathsf{X})$ in a reference configuration at time t=0, occupies the place

$$ {}^t\mathbf{x} = {}^t\chi(\mathsf{X}) = \chi\left({}^0\chi^{-1}({}^0\mathbf{x}), t\right) = \phi\left({}^0\mathbf{x}, t\right) = {}^t\phi\left({}^0\mathbf{x}\right) = {}^0\mathbf{x} + {}^t_0\mathbf{u} $$

at the time t.

The mapping ${}^t\boldsymbol{\phi}$ is called deformation,

and ${}^t_0\mathbf{u}$ is the **displacement** of the particle.
Observing an infinitesimal vicinity of the particle in the course of motion,

$$ {}^t\mathbf{x} + d\,{}^t\mathbf{x} = {}^t\phi\left({}^0\mathbf{x} + d\,{}^0\mathbf{x}\right) \approx {}^t\phi\left({}^0\mathbf{x}\right) + \frac{\partial\,{}^t\phi}{\partial\,{}^0\mathbf{x}} \cdot d\,{}^0\mathbf{x} $$

leads to the definition of the **deformation gradient**,

$$ {}^t_0\mathbf{F} = \frac{\partial\,{}^t\phi}{\partial\,{}^0\mathbf{x}} = {}^0\nabla\ {}^t_0\mathbf{u} + \mathbf{I} = {}^t_0\mathbf{H} + \mathbf{I} $$

Various tensor-valued measures of the local deformation, commonly addressed as **strains**, can be derived from ${}^t_0\mathbf{F}$.

Strain Tensors

polar decomposition

$$ {}_0^t\mathbf{F} = {}_0^t\mathbf{R} \cdot {}_0^t\mathbf{U} = {}_0^t\mathbf{V} \cdot {}_0^t\mathbf{R} $$

$$ \text{with} \qquad {}_0^t\mathbf{R} \cdot {}_0^t\mathbf{R}^T = \mathbf{I} $$

right CAUCHY-GREEN Tensor

$$ {}_0^t\mathbf{C} = {}_0^t\mathbf{F}^T \cdot {}_0^t\mathbf{F} = {}_0^t\mathbf{U}^2 = \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \cdot \mathbf{H} $$

- GREEN's quadratic strain tensor

$$ \begin{aligned} {}_0^t\mathbf{E}^{(\mathrm{G})} &= \tfrac{1}{2}\left({}_0^t\mathbf{U}^2 - \mathbf{I} \right) \\ &= \tfrac{1}{2}\left[{}^0\nabla \, {}_0^t\mathbf{u} + \left({}^0\nabla \, {}_0^t\mathbf{u} \right)^T + \left({}^0\nabla \, {}_0^t\mathbf{u} \right) \cdot \left({}^0\nabla \, {}_0^t\mathbf{u} \right)^T \right] \\ &= {}_0^t\mathbf{E} + \tfrac{1}{2}\left({}^0\nabla \, {}_0^t\mathbf{u} \right) \cdot \left({}^0\nabla \, {}_0^t\mathbf{u} \right)^T \end{aligned} $$

${}_0^t\mathbf{E}$ strain tensor for small deformations

- HENCKY's logarithmic ("true") strain tensor

$$ {}_0^t\mathbf{E}^{(H)} = \ln\left({}_0^t\mathbf{U} \right) $$

tensor of deformation rates

$$ {}^t\mathbf{D} = \tfrac{1}{2}\left[{}^t\nabla \, {}^t\mathbf{v} + \left({}^t\nabla \, {}^t\mathbf{v} \right)^T \right] $$

$$ {}^t\mathbf{D} = {}_0^t\mathbf{F}^{-T} \cdot {}^t\dot{\mathbf{E}}^{(\mathrm{G})} \cdot {}_0^t\mathbf{F}^{-1} $$

Forces and Stresses

The concept describes the **action** of the outside world on the body B in motion and the **interaction** between the different parts P of the body.

(a) **external body force**

$$\mathbf{F}_b(\mathrm{P}) = \int_{\mathrm{P}} \mathbf{b}(\mathbf{x}, t)\rho\, dV, \quad \mathrm{P} \subset \mathrm{B}$$

over the part P of B in the configuration $\boldsymbol{\chi}$.

(b) **contact force**

$$\mathbf{F}_c(\mathrm{P}) = \int_{\partial\mathrm{P}} \mathbf{t}(\mathbf{x};\mathrm{P})\, dS$$

extended over the boundary ∂P of P .

(c) **total resultant force**

$$\mathbf{F}(\mathrm{P}) = \mathbf{F}_b(\mathrm{P}) + \mathbf{F}_c(\mathrm{P})$$

(d) **stress principle**

$$\mathbf{t}(\mathbf{x};\mathrm{P}) = \mathbf{t}(\mathbf{x},\mathbf{n})$$

where **n** is the exteriour unit normal vector at the point **x** on the boundary ∂P of P

This implies the existence of a **stress-tensor field S(x)** such that the **stress vector t(x,n)** may be expressed by

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = \mathbf{n} \cdot \mathbf{S}(\mathbf{x})$$

Principal Stresses and Invariants

"eigenvalue" problem $\mathbf{S} \cdot \mathbf{n} = \sigma\, \mathbf{n}$

condition $\det(\mathbf{S} - \sigma\, \mathbf{I}) = 0$

characteristic equation $\sigma^3 - J_1 \sigma^2 - J_2 \sigma - J_3 = 0$

invariants of stress tensor

$$J_1(\mathbf{S}) = \mathrm{tr}(\mathbf{S}) = \sigma_{ii}$$

$$J_2(\mathbf{S}) = \tfrac{1}{2}\left[\mathrm{tr}(\mathbf{S}^2) - \mathrm{tr}^2(\mathbf{S})\right] = \tfrac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}\right)$$

$$J_3(\mathbf{S}) = \det(\mathbf{S}) = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}$$

three real solutions ("eigenvalues") = principal stresses

$$\sigma_I \geq \sigma_{II} \geq \sigma_{III}$$

with corresponding principal directions $\mathbf{n}_I, \mathbf{n}_{II}, \mathbf{n}_{III}$

$$\mathbf{S} \cdot \mathbf{n}_i = \sigma_i\, \mathbf{n}_i \quad (i = I, II, III)$$

$$\mathbf{S} = \begin{pmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{pmatrix} \mathbf{n}_i \mathbf{n}_j$$

invariants in terms of principal stresses

$$J_1(\mathbf{S}) = \sigma_I + \sigma_{II} + \sigma_{III}$$

$$J_2(\mathbf{S}) = -\left(\sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_I\sigma_{III}\right)$$

$$J_3(\mathbf{S}) = \sigma_I\sigma_{II}\sigma_{III}$$

stress deviator $\mathbf{S}' = \mathbf{S} - \frac{1}{3}\sigma_{ii}\mathbf{I}$

hydrostatic stress $\sigma_h = \frac{1}{3}\sigma_{ii}$

invariants of stress deviator

$$J_1(\mathbf{S}') = 0$$

$$J_2(\mathbf{S}') = \tfrac{1}{2}\sigma'_{ij}\sigma'_{ij}$$

$$= \tfrac{1}{6}\Big[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 +$$

$$+\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2\Big]$$

$$J_3(\mathbf{S}') = \tfrac{1}{3}\sigma'_{ij}\sigma'_{jk}\sigma'_{ki} = \tfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki}$$

VON MISES equivalent (effective) stress

$$\boxed{\sigma_e = \sqrt{3J_2(\mathbf{S}')}}$$

VON MISES yield condition

$$\boxed{\sigma_e \le R_0}$$

upper limit of purely elastic deformation

beginning plastic deformation

R_0 = yield strength

plane stress

$$\mathbf{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{e}_i \mathbf{e}_j$$

principal stresses

$$\left.\begin{matrix} \sigma_I \\ \sigma_{II} \end{matrix}\right\} = \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)^2 \pm \sqrt{\tfrac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2}$$

direction of first (maximum) principal stress

$$\tan 2\varphi_0 = \frac{2\sigma_{xy}}{\sigma_{yy} - \sigma_{xx}}$$

maximum shear stress

$$\tau_{\max} = \sqrt{\tfrac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2} = \tfrac{1}{2}\left(\sigma_I - \sigma_{II}\right)$$

direction of maximum shear stress

$$\varphi_1 = \varphi_0 \pm \frac{\pi}{4}$$

VON MISES equivalent (effective) stress

$$\sigma_e = \sqrt{\sigma_{xx}^2 + \sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2}$$

$$= \sqrt{\sigma_I^2 + \sigma_{II}^2 - \sigma_I \sigma_{II}}$$

Balance Equations

conservation of mass

global: any finite part P of B

$$\dot{m}(\mathsf{P}) = \frac{d}{dt}\int_{\mathsf{P}} \rho \, dV = 0$$

local (equation of continuity)

$$\dot{\rho} + \rho \operatorname{div}\dot{\mathbf{x}} = 0$$

balance of momentum

global

$$\mathbf{F}(\mathsf{P}) = \frac{d}{dt}\int_{\mathsf{P}} \rho \, \dot{\mathbf{x}} \, dV$$

local (CAUCHY's law of motion)

$$\operatorname{div}\mathbf{S} + \rho\,\mathbf{b} = \rho\,\ddot{\mathbf{x}}$$

balance of moment of momentum

for non-polar media

global

$$\int_{\mathsf{P}} \mathbf{x}\times\mathbf{b}\rho \, dV + \int_{\partial\mathsf{P}} \mathbf{x}\times\mathbf{t} \, dS = \frac{d}{dt}\int_{\mathsf{P}} \rho \, \mathbf{x}\times\dot{\mathbf{x}} \, dV$$

local (symmetry of stress tensor)

$$\mathbf{S} = \mathbf{S}^T$$

General Principles
governing the mechanical behaviour of materials

1. material frame-indifference

Constitutive equations must be invariant under changes of frame reference. If a constitutive equation is satisfied for a process with a motion and a symmetric stress tensor given by

$$\mathbf{x} = \chi(X, t), \quad \mathbf{S} = \mathbf{S}(X, t)$$

then it must be satisfied also for the motion and stress tensor given by

$$\tilde{\mathbf{x}} = \tilde{\chi}(X, \tilde{t}) = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \chi(X, t),$$

$$\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(X, \tilde{t}) = \mathbf{Q}(t) \cdot \mathbf{S}(X, t) \cdot \mathbf{Q}^T(t)$$

$$\tilde{t} = t - \tau$$

2. determinism

The stress in a body is determined by the history of the motion of that body.

3. local action

In determining the stress at a given particle X, the motion outside an arbitrary neighbourhood of X may be disregarded.

Constitutive Equations

principle of determinism:

$$\mathbf{S}(X,t) = \underset{\tau=-\infty}{\overset{t}{\mathrm{f}}}\left\{\chi(y,\tau), X\right\} \qquad x,\ \ y \in B$$

transformation of time: $\tau = t - s$

$$\mathbf{S}(X,t) = \underset{s=0}{\overset{\infty}{\mathrm{f}}}\left\{\chi(y,t-s), X\right\}$$

principle of material frame-indifference:

$$\mathbf{Q}\cdot \mathrm{f} \cdot \mathbf{Q}^{T} = \mathrm{f}\left\{\mathbf{Q}\cdot\chi\right\}$$

$$\mathbf{S}(X,t) = \underset{s=0}{\overset{\infty}{\mathrm{f}}}\left\{\chi(y,t-s) - \chi(X,t-s)\right\}$$

principle of local action:

$$\Omega : \quad \left\|\mathbf{y}-\mathbf{x}\right\| \leq \delta\ ;\ \mathbf{x} = \chi(X),\ \mathbf{y} = \chi(y)$$

$$\mathbf{S}(X,t) = \underset{s=0}{\overset{\infty}{\mathrm{f}}}\left\{ {}_{-\infty}^{t-s}\mathbf{F}(X),\ {}^{t-s}\nabla\left({}_{-\infty}^{t-s}\mathbf{F}(X)\right), \ldots\right\}$$

"**simple materials**"

$$\boxed{\mathbf{S}(X,t) = \underset{s=0}{\overset{\infty}{\mathrm{f}}}\left\{ {}_{-\infty}^{t-s}\mathbf{F}(X)\right\}}$$

principle of fading memory:

the memory of a simple material fades in time

Hooke's Law of Elasticity

general: $\mathbf{S} = \overset{<4>}{\mathbf{C}} \cdot\cdot \mathbf{E}$, $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$\mathbf{E} = \left(\overset{<4>}{\mathbf{C}}\right)^{-1} \cdot\cdot \mathbf{S}$, $\varepsilon_{ij} = C^{-1}_{ijkl}\,\sigma_{kl}$

$\overset{<4>}{\mathbf{C}}$ stiffness tensor (4th order)

$\left(\overset{<4>}{\mathbf{C}}\right)^{-1}$ compliance tensor (4th order)

isotropic: $\boxed{C_{ijkl} = 2G\delta_{ik}\delta_{jl} + \left(K - \tfrac{2}{3}G\right)\delta_{ij}\delta_{kl}}$

or

$$\mathbf{S} = 2G\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\mathbf{I}\right]$$

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}(\varepsilon_{kk})\,\delta_{ij}\right]$$

$$\mathbf{E} = \frac{1}{2G}\left[\mathbf{S} - \frac{\nu}{1+\nu}(\mathrm{tr}\,\mathbf{S})\mathbf{I}\right]$$

$$\varepsilon_{ij} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}(\sigma_{kk})\,\delta_{ij}\right]$$

G = shear modulus
K = bulk modulus
ν = POISSON's ratio

Material Parameters for Linear Elasticity
Elastizitätskonstanten

	$\lambda =$	$\mu =$	$E =$	$\nu =$	$K =$	$G =$
λ, μ	λ	μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\lambda+\frac{2}{3}\mu$	μ
G, K	$K-\frac{2}{3}G$	G	$\frac{3K\cdot G}{3K+G}$	$\frac{3K-2G}{6K+2G}$	K	G
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$

λ, μ	LAMÉ's coefficients	LAMÉsche Konstanten
G	shear modulus	Schubmodul
K	bulk modulus	Kompressionsmodul
E	YOUNG's modulus	Elastizitätsmodul
ν	POISSON's ratio	Querkontraktionszahl

Variational Principles

variational principles in mechanics

- Ø replace the (differential) equations of motion or equilibrium, as there are
  - • balance of momentum,
  - • balance of angular momentum,
  - • CAUCHY's field equations;
- Ø are extremum principles for energy type quantities, like
  - • work,
  - • kinetic energy,
  - • potential energy.

Differential equations of motion can be established by methods of variational calculus.

Variational Calculus

problem: find a set of functions $x_i(t)$, $i = 1, ..., n$ for which the integral

$$I = \int_{t_0}^{t_1} F\left(t, x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n\right) dt$$

becomes an extremum under given "boundary" conditions

$$x_i(t_0) = x_i^0 \; ; \quad x_i(t_1) = x_i^1 \qquad \text{(b.c.)}$$

definition of "varied" functions

$$\bar{x}_i(t) = x_i(t) + \varepsilon \xi_i(t) \text{ with } \xi_i(t_0) = \xi_i(t_1) = 0$$

$\xi_i(t)$ are arbitrary, differentiable functions meeting the b.c.

$$\Rightarrow \quad I(\varepsilon) = \int_{t_0}^{t_1} F\left(t, x_1 + \varepsilon\xi_1, ..., \dot{x}_1 + \varepsilon\dot{\xi}_1, ...\right) dt$$

the condition for *I* becoming an extremum

is

$$\boxed{\left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0} = 0}$$

$\delta x_i = \xi_i(t) \; ; \quad \delta \dot{x}_i = \dot{\xi}_i(t)$ are variations of x_i ; $\dot{x}_i$

$\delta I = \left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0}$ is the (first) variation of *I*

$\delta I = 0$	is the variational problem

the **variational problem** leads to

$$\delta I = \left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0} = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x_i}\xi_i + \frac{\partial F}{\partial \dot{x}_i}\dot{\xi}_i \right) dt = 0$$

partial integration yields

$$\int_{t_0}^{t_1} \left(\dot{\xi}_i \frac{\partial F}{\partial \dot{x}_i} \right) dt = \left[\xi_i \frac{\partial F}{\partial \dot{x}_i} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\xi_i \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} \right) dt$$

where [..] = 0 due to b.c.

$$\delta I = \int_{t_0}^{t_1} \xi_i \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} \right) dt = 0$$

and as $\xi_i(t)$ are arbitrary ("test functions")

⇒ EULER's differential equation of the variational problem

$$\boxed{\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} = 0}$$

Variational Problem in Mechanics

$\mathbf{u} = \mathbf{x}(t) - \mathbf{x}(t_0)$ displacement vektor

$\mathbf{w}$ arbitrary "virtual" displacement,

- independent of t
- $\mathbf{w}(t_0) = 0$ in the reference configuration

Z energy functional

$$\delta Z = \lim_{\varepsilon \to 0} \left(\frac{Z(\mathbf{u} + \varepsilon \mathbf{w}) - Z(\mathbf{u})}{\varepsilon} \right) = \left. \frac{\partial Z(\mathbf{u} + \varepsilon \mathbf{w})}{\partial \varepsilon} \right|_{\varepsilon = 0}$$

variation of Z at $\mathbf{u}$ in $\mathbf{w}$ direction
(GÂTEAUX derivative)

$$\delta \mathbf{u} = \delta \mathbf{x} = \left. \frac{\partial (\mathbf{u} + \varepsilon \mathbf{w})}{\partial \varepsilon} \right|_{\varepsilon = 0} = \mathbf{w} \qquad \text{virtual displacement}$$

(1) $\delta Z(\mathbf{u}, \alpha \mathbf{w}) = \alpha \delta Z(\mathbf{u}, \mathbf{w})$

(2) $\delta Z(\mathbf{u}, \mathbf{w}_1 + \mathbf{w}_2) = \delta Z(\mathbf{u}, \mathbf{w}_1) + \delta Z(\mathbf{u}, \mathbf{w}_2)$

example: $Z = \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \mathbf{b} \cdot \mathbf{u}$

$$\delta Z = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\tfrac{1}{2} (\mathbf{u} + \varepsilon \mathbf{w}) \cdot (\mathbf{u} + \varepsilon \mathbf{w}) - \mathbf{b} \cdot (\mathbf{u} + \varepsilon \mathbf{w}) - \tfrac{1}{2} \mathbf{u} \cdot \mathbf{u} + \mathbf{b} \cdot \mathbf{u} \right]$$

$$= \lim_{\varepsilon \to 0} \left[\mathbf{u} \cdot \mathbf{w} + \tfrac{1}{2} \varepsilon \mathbf{w} \cdot \mathbf{w} - \mathbf{b} \cdot \mathbf{w} \right]$$

$$= (\mathbf{u} - \mathbf{b}) \cdot \delta \mathbf{u} \qquad \delta Z \text{ is linear in } \delta \mathbf{u}$$

Principle of Virtual Work

CAUCHY's field equations of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho \ddot{u}_j \quad , \quad j = 1, 2, 3$$

multiply by virtual displacement δu_j and integrate over the volume V:

$$\int_V \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j \, dV + \int_V \rho b_j \delta u_j \, dV = \int_V \rho \ddot{u}_j \delta u_j \, dV$$

$$\int_V \frac{\partial \sigma_{ij}}{\partial x_i} \delta u_j \, dV = \int_V \frac{\partial}{\partial x_i} \left(\sigma_{ij} \, \delta u_j \right) dV - \int_V \sigma_{ij} \, \frac{\partial \left(\delta u_j \right)}{\partial x_i} dV$$

GAUß' theorem

$$\int_V \frac{\partial}{\partial x_i} \left(\sigma_{ij} \, \delta u_j \right) dV = \int_{\partial V} n_i \, \sigma_{ij} \, \delta u_j \, dA = \int_{\partial V} t_j \, \delta u_j \, dA$$

$$\int_V \sigma_{ij} \, \frac{\partial \left(\delta u_j \right)}{\partial x_i} dV = \int_V \sigma_{ij} \, \delta \frac{\partial u_j}{\partial x_i} dV = \int_V \sigma_{ij} \, \delta \varepsilon_{ij} \, dV$$

$$\int_V \rho \ddot{u}_j \delta u_j \, dV = \frac{d}{dt} \int_V \rho \dot{u}_j \delta u_j \, dV - \delta \int_V \tfrac{1}{2} \rho \dot{u}_j \dot{u}_j \, dV$$

$$\delta A - \delta W = \delta P - \delta E$$

virtual work of external forces

$$\delta A = \int_{\partial V} t_j \, \delta u_j \, dA + \int_V \rho b_j \delta u_j \, dV$$

virtual work of stresses (virtual strain energy)

$$\delta W = \int_V \sigma_{ij} \, \delta \varepsilon_{ij} \, dV$$

virtual power of momentum

$$\delta P = \frac{d}{dt} \int_V \rho \, \dot{u}_j \delta u_j \, dV$$

virtual kinetic energy

$$\delta E = \delta \int_V \tfrac{1}{2} \rho \, \dot{u}_j \dot{u}_j \, dV$$

virtual work of mass acceleration

$$\delta B = \delta P - \delta E = \int_V \rho \, \ddot{u}_j \delta u_j \, dV$$

$$\delta(A - W - P + E) = 0$$

The variation of the energy functional (A - W - P + E) vanishes.

or

The energy functional (A - W - P + E) becomes an extremum (minimum) among all admissible states (virtual displacements).

special cases:

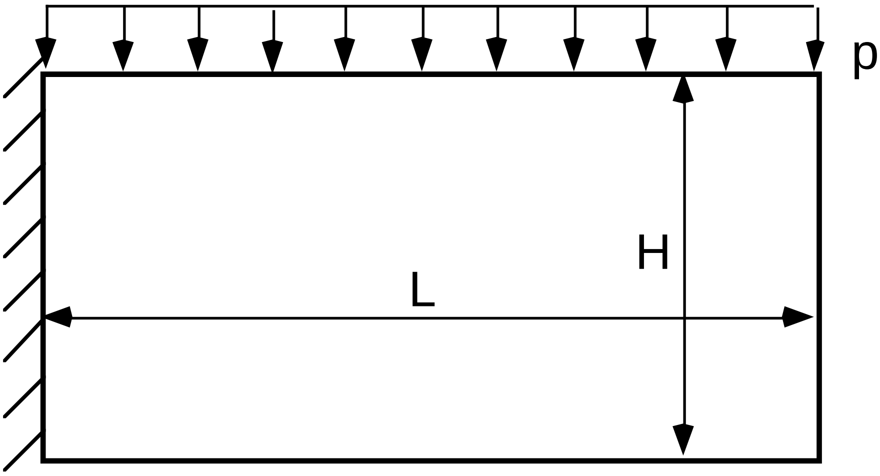
- **rigid body** $\delta W = 0$
- **elastic body** $\delta W = \int_V C_{ijkl}\, \varepsilon_{kl} \delta\varepsilon_{ij}\, dV$

$W = \frac{1}{2}\int_V C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}\, dV$ elastic strain energy

- **static problem (equilibrium)**

$$\delta B = \delta P - \delta E = 0$$

Example: plane panel



thickness = 1

$$\{t\} = \begin{pmatrix} t_x(x,y) \\ t_y(x,y) \end{pmatrix}_{\text{boundary}} = \begin{pmatrix} 0 \\ -p \end{pmatrix}_{0 \le x \le L, y = H}$$

displacement field $\{u\} = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix}$

boundary conditions (b.c.) $u_x(0,y) = u_y(0,y) = 0$

dimensionless coordinates $\xi = \frac{x}{L}; \quad \eta = \frac{y}{H}$

global shape functions, $\varphi_i(\xi,\eta)$, fulfilling b. c.

$$\boxed{\begin{array}{lll} \varphi_1 = \xi, & \varphi_2 = \xi^2, & \varphi_3 = \xi\eta \\ \varphi_4 = \xi^3, & \varphi_5 = \xi^2\eta, & \varphi_6 = \xi\eta^2 \end{array}}$$

$$\{\Phi\} = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 \end{pmatrix}$$

series expansion of displacement field

$$\{\mathrm{u}\} = \begin{pmatrix} \sum_{i=1}^{6} \alpha_i \varphi_i \\ \sum_{i=1}^{6} \alpha_{i+6} \varphi_i \end{pmatrix} = \{\Phi\}\{\alpha\} \qquad \text{with} \qquad \{\alpha\} = \begin{pmatrix} \alpha_1 \\ . \\ . \\ \alpha_{12} \end{pmatrix}$$

$\{\alpha\}$ is (12×1) matrix of unknowns

strain matrix

$$\{\varepsilon\} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{pmatrix}$$

$$= \{\mathrm{D}\}\{\mathrm{u}\} = \{\mathrm{D}\}\{\Phi\}\{\alpha\} = \{\mathrm{B}\}\{\alpha\}$$

differential operator

$$\{\mathrm{D}\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{L\partial \xi} & 0 \\ 0 & \frac{\partial}{H\partial \eta} \\ \frac{\partial}{H\partial \eta} & \frac{\partial}{L\partial \xi} \end{pmatrix}$$

$$\{\mathrm{B}\} = \begin{pmatrix} \frac{1}{L} & \frac{\xi}{L} & \frac{2\eta}{L} & \frac{3\xi^2}{L} & \frac{2\xi\eta}{L} & \frac{\eta^2}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\xi}{H} & 0 & \frac{\xi^2}{H} & \frac{2\xi\eta}{H} \\ 0 & 0 & \frac{\xi}{H} & 0 & \frac{\xi^2}{H} & \frac{2\xi\eta}{H} & \frac{1}{L} & \frac{2\xi}{L} & \frac{\eta}{L} & \frac{3\xi^2}{L} & \frac{2\xi\eta}{L} & \frac{\eta^2}{L} \end{pmatrix}$$

stress matrix and HOOKE's law

$$\{\sigma\} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \{\mathrm{C}\}\{\varepsilon\}$$

$\{\mathrm{C}\}$ stiffness matrix

for plane stress: $\sigma_{zz} = 0$

$$\{\mathrm{C}\} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{pmatrix} = \{\mathrm{C}\}^T$$

for plane strain: $\varepsilon_{zz} = 0$

$$\{\mathrm{C}\} = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{pmatrix} = \{\mathrm{C}\}^T$$

virtual work of external forces

$$\delta A = \int_{\xi=0}^{1} \{\mathrm{t}\}^T [\delta\{\mathrm{u}\}]_{\eta=1} L d\xi = \{\mathrm{f}\}^T \delta\{\alpha\}$$

virtual work of stresses

$$\delta W = \int_{\xi=0}^{1} \int_{\eta=0}^{1} \{\sigma\}^T \delta\{\varepsilon\} L d\xi \, H d\eta = \{\alpha\}^T \{\mathrm{K}\} \delta\{\alpha\}$$

"generalized" forces

$$\{\mathrm{f}\}^T = \int_{\xi=0}^{1} \{\mathrm{t}(\xi)\}^T \{\Phi(\xi)\}_{\eta=1} L d\xi = L \{\mathrm{t}\}^T \int_{\xi=0}^{1} \{\Phi(\xi)\}_{\eta=1} d\xi$$

$$= -pL \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{3} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{3} & \tfrac{1}{2} \end{pmatrix}$$

global stiffness matrix

$$\{\mathrm{K}\} = LH \int_{\xi=0}^{1} \int_{\eta=0}^{1} \{\mathrm{B}\}^T \{\mathrm{C}\} \{\mathrm{B}\} d\xi \, d\eta = \{\mathrm{K}\}^T$$

principle of virtual work

$$\delta A - \delta W = \left(\{\mathrm{f}\}^T - \{\alpha\}^T \{\mathrm{K}\}\right) \delta\{\alpha\}$$

as $\delta\{\alpha\}$ is arbitrary

$$\boxed{\{\mathrm{K}\}\{\alpha\} = \{\mathrm{f}\}}$$

linear system of equations for unknowns α_i

FE Modelling and Computational Procedure

Finite element methods (FEM) base on variational principles for minimizing some potential like, e.g., the potential energy of a mechanical systems.

Variational methods replace the solution of the corresponding boundary value problem.

For the so-called "deformation methods", FEM is based upon the

Principle of Virtual Work

$$\boxed{\delta\Pi = -\int_{\mathrm{B}} \mathbf{S}\cdot\cdot\,\delta(\mathrm{grad}\,\mathbf{u})\,dV + \int_{\mathrm{B}} \rho\mathbf{b}\cdot\delta\mathbf{u}\,dV + \int_{\partial\mathrm{B}} \mathbf{t}\cdot\delta\mathbf{u}\,dA = 0}$$

- stress-tensor field $\mathbf{T}(\mathbf{x})$,
- external body forces, $\mathbf{b}(\mathbf{x})$, defined for any $x \in \mathrm{B}$,
- contact forces or tractions $\mathbf{t}(\mathbf{x})$, defined for any x on the boundary $\partial\mathrm{B}$.

- The continuous body B is separated by imaginary lines or surfaces into a number, K, of (finite) elements B_k.

  The union

$$\tilde{\mathrm{B}} = \bigcup_{k=1}^{K} \mathrm{B}_k$$

  is the finite model of the body.

- The elements are assumed to be interconnected at a discrete number, N, of nodal points, $\mathbf{x}_n$, situated on their boundaries.

  The displacements of these nodal points

$$\mathbf{u}_n = \mathbf{u}(\mathbf{x}_n), \quad n = 1,\ldots N$$

  are the basic unknown parameters of the problem.

- A set of functions, $\varphi_i^{(k)}(\xi)$, $i = 1, \ldots N_k$, $\boldsymbol{\xi}$ being local coordinates, is chosen to define uniquely the state of displacement, $\tilde{\mathbf{u}}^{(k)}(\xi)$, within each element, *k*, in terms of its nodal displacements, $\mathbf{u}_i^{(k)}$,

$$\tilde{\mathbf{u}}^{(k)}(\xi) = \sum_{i=1}^{N_k} \varphi_i^{(k)}(\xi)\,\mathbf{u}_i^{(k)}$$

with $\mathbf{u}_i^{(k)} = \tilde{\mathbf{u}}^{(k)}\left(\xi_i^{(k)}\right), \quad \varphi_i^{(k)}\left(\xi_j^{(k)}\right) = \delta_{ij}$.

- The displacement functions uniquely define the state of deformation, i.e. some strain tensor, $\tilde{\mathbf{E}}^{(k)}(\xi)$, within each element in terms of the nodal displacements, e.g., small strain

$$\tilde{\mathbf{E}}^{(k)}(\xi) = \frac{1}{2}\sum_{i=1}^{N_k}\left[\frac{\partial \varphi_i^{(k)}}{\partial \xi}\cdot\frac{\partial \xi}{\partial \mathbf{x}}\cdot\mathbf{u}_i^{(k)} + \left(\frac{\partial \varphi_i^{(k)}}{\partial \xi}\cdot\frac{\partial \xi}{\partial \mathbf{x}}\cdot\mathbf{u}_i^{(k)}\right)^T\right],$$

with $\dfrac{\partial \xi}{\partial \mathbf{x}}$ = JACOBIAN matrix.

- These strains, together with the constitutive properties of the material, determine the state of stress, $\tilde{\mathbf{S}}^{(k)}(\xi)$, throughout the element and also on its boundaries.

- A system of forces, $\mathbf{t}_i^{(k)}$ concentrated at the nodes equilibrating the boundary stresses, $\tilde{\mathbf{t}}^{(k)} = \mathbf{n}^{(k)}\cdot\tilde{\mathbf{S}}^{(k)}$, is determined, resulting in a force-displacement or "stiffness" relationship for each element.

$$\mathbf{t}_i^{(k)} = \sum_{i=1}^{N_k}\mathbf{C}_{ij}^{(k)}\,\mathbf{u}_j^{(k)}.$$

- Nodal displacements, $\mathbf{u}_i^{(k)}$, nodal forces, $\mathbf{t}_i^{(k)}$, and element stiffnesses, $\mathbf{C}_{ij}^{(k)}$, are

  assembled according to the conditions of connectivity, $\mathbf{u}_i^{(k)} = \sum_{n=1}^{N} A_{in}^{(k)}\,\mathbf{u}_n$,

  for all elements to compose the system of equations ensuring the conditions of compatibility and equilibrium throughout.

$$\tilde{\mathbf{u}}(\mathbf{x}) = \sum_{k=1}^{K}\tilde{\mathbf{u}}^{(k)}(\mathbf{x}) = \sum_{n=1}^{N}\psi_n(\mathbf{x})\,\mathbf{u}_n$$

where $\tilde{\mathbf{u}}^{(k)}(\mathbf{x}) = \begin{cases}\sum_{i=1}^{N_k}\varphi_i^{(k)}\,\mathbf{u}_i^{(k)} & \mathbf{x}\in \mathrm{B}_k\\ 0 & \text{else}\end{cases}, \qquad \psi_n(\mathbf{x}) = \sum_{k=1}^{K}\sum_{i=1}^{N_k}\varphi_i^{(k)}A_{in}^{(k)}$

- Any system of nodal displacements listed for the whole structure in which all the elements participate, automatically satisfies the condition of compatibility.

As the equilibrium condition has already been satisfied within each element all that is necessary is to establish equilibrium at the nodes of the structure. This is done by the principle of virtual work.

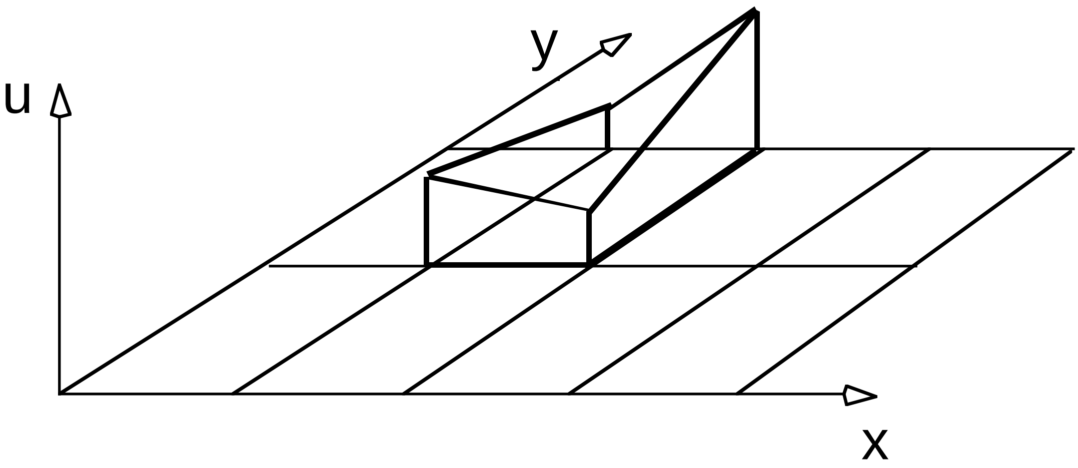
- The resulting equations governing the mechanical behaviour of the entire structure contain the nodal displacements as unknowns.

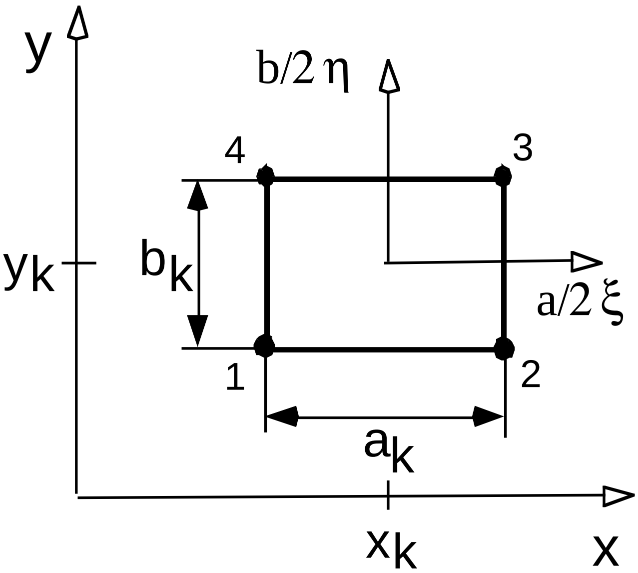
$$\boxed{\mathbf{s}_n + \mathbf{f}_n + \mathbf{p}_n = \mathbf{0}\,, \quad n = 1,\ldots,N}$$

$$\boxed{\mathbf{s}_n = -\int_{\mathrm{B}} \mathbf{S} \cdot \frac{\partial \psi_n}{\partial \mathbf{x}}\, dV\,; \quad \mathbf{f}_n = \int_{\mathrm{B}} \rho \mathbf{b}\, \psi_n\, dV\,; \quad \mathbf{p}_n = \int_{\partial \mathrm{B}} \mathbf{t}\, \psi_n\, dA}$$

A solution of this system of equations provides an approximate solution of the fields of displacements, strains and stresses throughout the domain of the body.

Shape Functions for Orthogonal Elements





element (k):

global coordinates

$$x_k - \tfrac{1}{2}a_k \le x \le x_k + \tfrac{1}{2}a_k$$

$$y_k - \tfrac{1}{2}b_k \le y \le y_k + \tfrac{1}{2}b_k$$

local coordinates: $-1 \le \xi, \eta \le +1$

$$\xi = \frac{2}{a_k}\left(x - x_k\right) \quad , \quad \eta = \frac{2}{b_k}\left(y - y_k\right)$$

4-Node Elements

"linear" plane elements: $N_k = 4$

shape functions $\varphi_i(\xi_j, \eta_j) = \delta_{ij}$ $\quad i, j = 1, ..., 4$

$$\boxed{\begin{aligned} \varphi_1(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ \varphi_2(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ \varphi_3(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ \varphi_4(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}}$$

displacement field: $\{\mathrm{u}\} = \{\Phi\}\{\hat{\mathrm{u}}\}_k$

$$\begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 \end{pmatrix} \begin{pmatrix} u_{1x}^{(k)} \\ u_{1y}^{(k)} \\ .. \\ .. \\ u_{4x}^{(k)} \\ u_{4y}^{(k)} \end{pmatrix}$$

strain matrix

$$\{\varepsilon\} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \{\mathrm{D}\}\{\mathrm{u}\} = \{\mathrm{D}\}\{\Phi\}\{\hat{\mathrm{u}}\}_k = \{\mathrm{B}\}\{\hat{\mathrm{u}}\}_k$$

$\{\hat{\mathrm{u}}\}_k$ nodal displacement matrix of element (k)

$$\text{differential operator } \{D\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{2}{a_k}\frac{\partial}{\partial \xi} & 0 \\ 0 & \frac{2}{b_k}\frac{\partial}{\partial \eta} \\ \frac{2}{b_k}\frac{\partial}{\partial \eta} & \frac{2}{a_k}\frac{\partial}{\partial \xi} \end{pmatrix}$$

$$\{\hat{u}\}_k = \{A\}_k \{\hat{u}\} = \{A\}_k \begin{pmatrix} u_{1x} \\ u_{1y} \\ .. \\ .. \\ u_{Nx} \\ u_{Ny} \end{pmatrix}$$

$\{\hat{u}\}$	global nodal displacement matrix
$\{A\}_k$	$(8 \times 2N)$ incidence or connectivity matrix
N =	total number of nodes
$2N$ =	number of degrees of freedom

$$\{\varepsilon\} = \{B\}\{A\}_k \{\hat{u}\}$$

stress matrix and HOOKE's law

$$\{\sigma\} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \{C\}_k \{\varepsilon\} = \{C\}_k \{B\}\{A\}_k \{\hat{u}\}$$

principle of virtual work

$$\delta A - \delta W = 0$$

virtual work of external forces

$$\delta A = \int_{\partial S} \{\mathrm{t}\}^T \delta\{\mathrm{u}\}\, ds = \sum_{k=1}^{K} \int_{\partial S_k} \{\mathrm{t}\}^T \{\Phi\}\, ds\, \delta\{\hat{\mathrm{u}}\}_k = \{\hat{\mathrm{f}}\}^T \delta\{\hat{\mathrm{u}}\}$$

nodal forces $\{\hat{\mathrm{f}}\} = \sum_{k=1}^{K} \{\mathrm{A}\}_k^T \int_{\partial S_k} \{\Phi\}^T \{\mathrm{t}(s)\}\, ds$

virtual work of stresses

$$\delta W = \sum_{k=1}^{K} \int_{x=-\frac{1}{2}a_k}^{+\frac{1}{2}a_k} \int_{y=-\frac{1}{2}b_k}^{+\frac{1}{2}b_k} \{\sigma\}^T \delta\{\varepsilon\}\, dx\, dy = \{\hat{\mathrm{u}}\}^T \{\mathrm{K}\}\, \delta\{\hat{\mathrm{u}}\}$$

element stiffness matrix

$$\{\mathrm{K}\}_k = \frac{a_k b_k}{4} \int_{\xi=-1}^{+1} \int_{\eta=-1}^{+1} \{\mathrm{B}\}^T \{\mathrm{C}\}_k \{\mathrm{B}\}\, d\xi\, d\eta = \{\mathrm{K}\}_k^T$$

global stiffness matrix

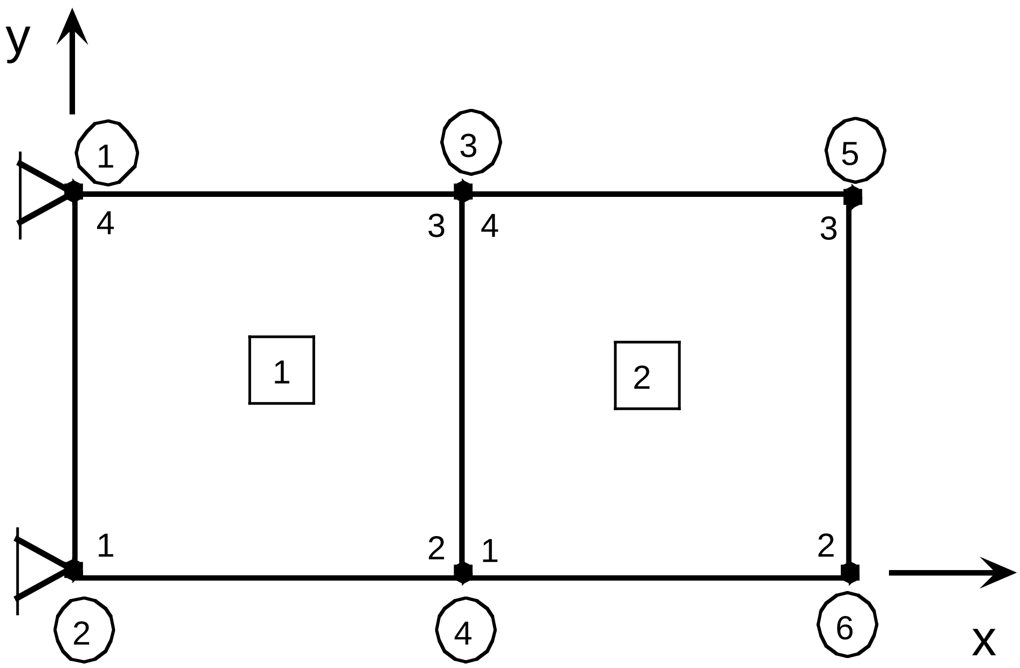
$$\{\mathrm{K}\} = \sum_{k=1}^{K} \{\mathrm{A}\}_k^T \{\mathrm{K}\}_k \{\mathrm{A}\}_k = \{\mathrm{K}\}^T$$

$$\delta A - \delta W = \left(\{\hat{\mathrm{f}}\}^T - \{\hat{\mathrm{u}}\}^T \{\mathrm{K}\}\right) \delta\{\hat{\mathrm{u}}\} = 0$$

as $\delta\{\hat{\mathrm{u}}\}$ is arbitrary

$$\boxed{\{\mathrm{K}\}\{\hat{\mathrm{u}}\} = \{\hat{\mathrm{f}}\}}$$

linear system of equations for $2N$ unknowns $\{\hat{\mathrm{u}}\}$



	global nodes	(1), ..., (6)			
	elements	[1], [2]			
nodal coordinates	(1):	0	,	H	
	(2):	0	,	0	
	(3):	L/2	,	H	
	(4):	L/2	,	0	
	(5):	L	,	H	
	(6):	L	,	0	
elements	[1]:	(2)	(4)	(3)	(1)
	[2]:	(4)	(6)	(5)	(3)

$$a_k = L/2\,, \quad b_k = H \quad , \quad k = 1,2$$

displacement b.c.: $u_{1x} = u_{1y} = u_{2x} = u_{2y} = 0$

nodal displacements

$$\{\hat{u}\}_k = \begin{pmatrix} u_{1x}^{(k)} \\ u_{1y}^{(k)} \\ u_{2x}^{(k)} \\ u_{2y}^{(k)} \\ u_{3x}^{(k)} \\ u_{3y}^{(k)} \\ u_{4x}^{(k)} \\ u_{4y}^{(k)} \end{pmatrix}, \qquad \{\hat{u}\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \\ u_{6y} \end{pmatrix} \quad \{\hat{u}\}_k = \{A\}_k \{\hat{u}\}$$

$$\{A\}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

element stiffness matrix (k = 1, 2)

$$\{\mathrm{K}\}_k = \frac{LH}{8}\int_{\xi=-1}^{+1}\int_{\eta=-1}^{+1}\{\mathrm{B}\}^T\{\mathrm{C}\}_k\{\mathrm{B}\}d\xi\,d\eta = \{\mathrm{K}\}_k^T$$

global stiffness matrix $\{\mathrm{K}\} = \sum_{k=1}^{2}\{\mathrm{A}\}_k^T\{\mathrm{K}\}_k\{\mathrm{A}\}_k = \{\mathrm{K}\}^T$

element nodal forces $\{\hat{\mathrm{f}}\}_k = \int_{\partial \mathrm{B}_k}\{\Phi\}^T\{\mathrm{t}(s)\}_k\,ds$

pressure load $\{\mathrm{t}\}_k = \begin{pmatrix} 0 \\ -p(\xi) \end{pmatrix}$

$$\{\hat{\mathrm{f}}\}_k = \frac{L}{4}\int_{\xi=-1}^{+1}\{\Phi(\xi)\}^T\Big|_{\eta=1}\{\mathrm{t}(\xi)\}_k\,d\xi + \{\hat{\mathrm{r}}\}_k$$

or interpolation for arbitrary function $\{\mathrm{t}(\xi)\}$

$\{\mathrm{t}(\xi)\}\big|_{\eta=1} = \{\Phi\}\big|_{\eta=1}\{\hat{\mathrm{t}}\}_k$, $\{\hat{\mathrm{t}}\}_k$ = (8×1) matrix

$$\{\hat{\mathrm{f}}\}_k = \frac{L}{4}\left[\int_{\xi=-1}^{+1}\left(\{\Phi\}^T\{\Phi\}\right)\Big|_{\eta=1}d\xi\right]\{\hat{\mathrm{t}}\}_k + \{\hat{\mathrm{r}}\}_k$$

reaction forces (k = 1, only)

$$\{\hat{\mathrm{r}}\}_1^T = \begin{pmatrix} r_{1x} & r_{1y} & 0 & 0 & 0 & 0 & r_{4x} & r_{4y} \end{pmatrix}$$

global nodal forces $\{\hat{\mathrm{f}}\} = \sum_{k=1}^{2}\{\mathrm{A}\}_k^T\{\hat{\mathrm{f}}\}_k$

system of equations

$$\begin{pmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & K_{1,5} & K_{1,6} & K_{1,7} & K_{1,8} & 0 & 0 & 0 & 0 \\ K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} & K_{2,5} & K_{2,6} & K_{2,7} & K_{2,8} & 0 & 0 & 0 & 0 \\ K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} & K_{3,5} & K_{3,6} & K_{3,7} & K_{3,8} & 0 & 0 & 0 & 0 \\ K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} & K_{4,5} & K_{4,6} & K_{4,7} & K_{4,8} & 0 & 0 & 0 & 0 \\ K_{5,1} & K_{5,2} & K_{5,3} & K_{5,4} & K_{5,5} & K_{5,6} & K_{5,7} & K_{5,8} & K_{5,9} & K_{5,10} & K_{5,11} & K_{5,12} \\ K_{6,1} & K_{6,2} & K_{6,3} & K_{6,4} & K_{6,5} & K_{6,6} & K_{6,7} & K_{6,8} & K_{6,9} & K_{6,10} & K_{6,11} & K_{6,12} \\ K_{7,1} & K_{7,2} & K_{7,3} & K_{7,4} & K_{7,5} & K_{7,6} & K_{7,7} & K_{7,8} & K_{7,9} & K_{7,10} & K_{7,11} & K_{7,12} \\ K_{8,1} & K_{8,2} & K_{8,3} & K_{8,4} & K_{8,5} & K_{8,6} & K_{8,7} & K_{8,8} & K_{8,9} & K_{8,10} & K_{8,11} & K_{8,12} \\ 0 & 0 & 0 & 0 & K_{9,5} & K_{9,6} & K_{9,7} & K_{9,8} & K_{9,9} & K_{9,10} & K_{9,11} & K_{9,12} \\ 0 & 0 & 0 & 0 & K_{10,5} & K_{10,6} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} & K_{10,11} & K_{10,12} \\ 0 & 0 & 0 & 0 & K_{11,5} & K_{11,6} & K_{12,7} & K_{11,8} & K_{11,9} & K_{11,10} & K_{11,11} & K_{11,12} \\ 0 & 0 & 0 & 0 & K_{12,5} & K_{12,6} & K_{12,7} & K_{12,8} & K_{12,9} & K_{12,10} & K_{12,11} & K_{12,12} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \\ u_{6y} \end{pmatrix} = \begin{pmatrix} 0 \\ f_{1y} \\ 0 \\ 0 \\ 0 \\ f_{3y} \\ 0 \\ 0 \\ 0 \\ f_{5y} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} r_{1x} \\ r_{1y} \\ r_{2x} \\ r_{2y} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

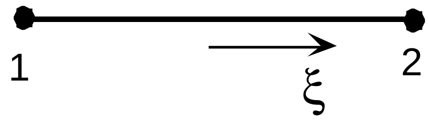
Lagrangian Interpolation Polynomials

$$g_i^{(n)}(\xi) = \prod_{\substack{j=1,\\ j\neq i}}^{n+1} \frac{\xi_j - \xi}{\xi_j - \xi_i}$$

n = order, i = 1, ..., n+1

1D: "truss" element

n = 1 (linear):

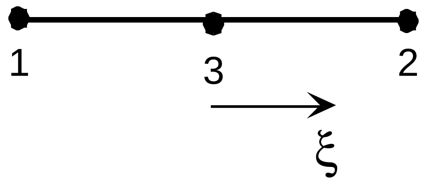


2 nodes $\xi_1 = -1\,; \quad \xi_2 = +1$

i = 1: $\varphi_1(\xi) = g_1^{(1)}(\xi) = \tfrac{1}{2}(1-\xi)$

i = 2: $\varphi_2(\xi) = g_2^{(1)}(\xi) = \tfrac{1}{2}(1+\xi)$

n = 2 (quadratic):



3 nodes $\xi_1 = -1\,; \quad \xi_2 = +1\,; \quad \xi_3 = 0$ 1

i=1: $\varphi_1(\xi) = g_1^{(2)}(\xi) = -\tfrac{1}{2}\xi(1-\xi) = \tfrac{1}{2}(1-\xi) - \tfrac{1}{2}(1-\xi^2)$

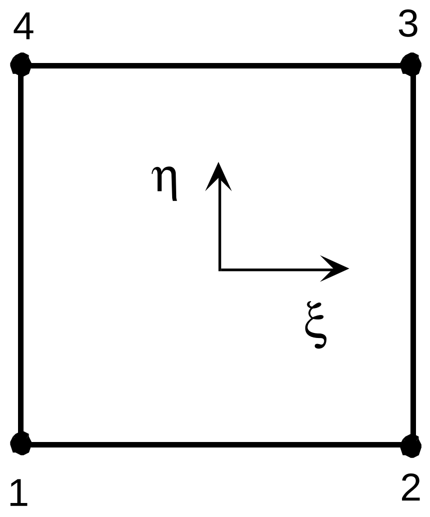
i=2: $\varphi_2(\xi) = g_2^{(2)}(\xi) = +\tfrac{1}{2}(1+\xi)\xi = \tfrac{1}{2}(1+\xi) - \tfrac{1}{2}(1-\xi^2)$

i=3: $\varphi_3(\xi) = g_3^{(2)}(\xi) = (1+\xi)(1-\xi) = 1-\xi^2$

$$\{\Phi\} = (\varphi_1 \quad \varphi_2 \quad \varphi_3)$$

2D: quadrilateral element

n = 1 (linear) 4 nodes



$$\xi_1 = -1 \quad \eta_1 = -1$$
$$\xi_2 = +1 \quad \eta_2 = -1$$
$$\xi_3 = +1 \quad \eta_3 = +1$$
$$\xi_4 = -1 \quad \eta_4 = +1$$

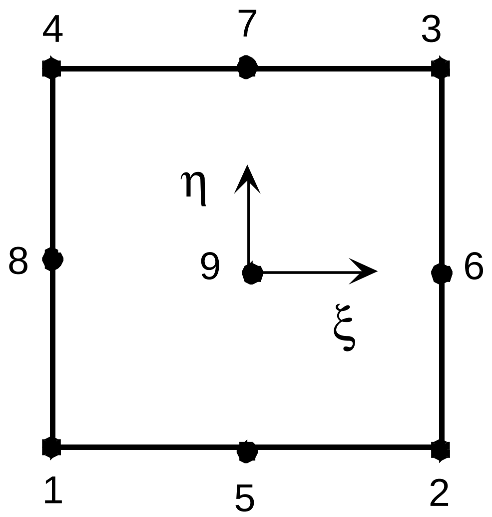
$$\begin{pmatrix} \varphi_1(\xi,\eta) & \varphi_3(\xi,\eta) \\ \varphi_2(\xi,\eta) & \varphi_4(\xi,\eta) \end{pmatrix} = \begin{pmatrix} g_1^{(1)}(\xi) g_1^{(1)}(\eta) & g_1^{(1)}(\xi) g_2^{(1)}(\eta) \\ g_2^{(1)}(\xi) g_1^{(1)}(\eta) & g_2^{(1)}(\xi) g_2^{(1)}(\eta) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}(1-\xi)(1-\eta) & \frac{1}{4}(1-\xi)(1+\eta) \\ \frac{1}{4}(1+\xi)(1-\eta) & \frac{1}{4}(1+\xi)(1+\eta) \end{pmatrix}$$

$$\{\mathrm{u}\} = \{\Phi\}\{\hat{\mathrm{u}}\}_k$$

$$\{\Phi\} = \begin{pmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 \end{pmatrix}$$

n = 2 (quadratic): 9 nodes



$$\xi_5 = 0 \quad \eta_5 = -1$$
$$\xi_6 = +1 \quad \eta_6 = 0$$
$$\xi_7 = 0 \quad \eta_7 = +1$$
$$\xi_8 = -1 \quad \eta_8 = 0$$
$$\xi_9 = 0 \quad \eta_9 = 0$$

$$\begin{pmatrix} \varphi_1 & \varphi_3 & \varphi_8 \\ \varphi_2 & \varphi_4 & \varphi_6 \\ \varphi_5 & \varphi_7 & \varphi_9 \end{pmatrix} =$$

$$\begin{pmatrix} g_1^{(2)}(\xi)g_1^{(2)}(\eta) & g_1^{(2)}(\xi)g_2^{(2)}(\eta) & g_1^{(2)}(\xi)g_3^{(2)}(\eta) \\ g_2^{(2)}(\xi)g_1^{(2)}(\eta) & g_2^{(2)}(\xi)g_3^{(2)}(\eta) & g_2^{(2)}(\xi)g_3^{(2)}(\eta) \\ g_3^{(2)}(\xi)g_1^{(2)}(\eta) & g_3^{(2)}(\xi)g_2^{(2)}(\eta) & g_3^{(2)}(\xi)g_3^{(2)}(\eta) \end{pmatrix}$$

$$\{\Phi\} = \begin{pmatrix} \varphi_1 & 0 & \varphi_2 & \dots & \dots & \dots & \varphi_9 & 0 \\ 0 & \varphi_1 & 0 & \dots & \dots & \dots & 0 & \varphi_9 \end{pmatrix}$$

3D: "brick" element

$$\varphi_1(\xi,\eta,\zeta) = g_1^{(n)}(\xi) g_1^{(n)}(\eta) g_1^{(n)}(\zeta)$$

$$\varphi_2(\xi,\eta,\zeta) = g_2^{(n)}(\xi) g_1^{(n)}(\eta) g_1^{(n)}(\zeta)$$

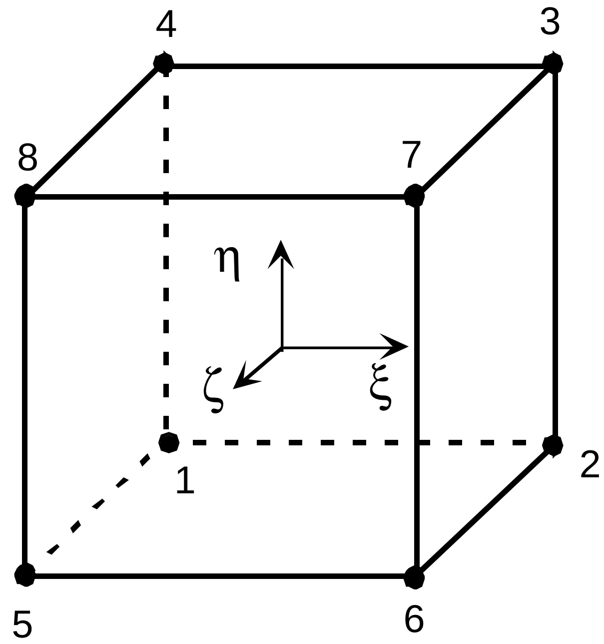
$$\varphi_3(\xi,\eta,\zeta) = g_2^{(n)}(\xi) g_2^{(n)}(\eta) g_1^{(n)}(\zeta)$$

$$\varphi_4(\xi,\eta,\zeta) = g_1^{(n)}(\xi) g_2^{(n)}(\eta) g_1^{(n)}(\zeta)$$

$$\varphi_5(\xi,\eta,\zeta) = g_1^{(n)}(\xi) g_1^{(n)}(\eta) g_2^{(n)}(\zeta)$$

etc.

n = 1 (linear) 8 nodes



$$\{\Phi\} = \begin{pmatrix} \varphi_1 & 0 & 0 & \ldots & \ldots & \ldots & \varphi_8 & 0 & 0 \\ 0 & \varphi_1 & 0 & \ldots & \ldots & \ldots & 0 & \varphi_8 & 0 \\ 0 & 0 & \varphi_1 & \ldots & \ldots & \ldots & 0 & 0 & \varphi_8 \end{pmatrix}$$

n = 2 (quadratic) 27 nodes

Note: for quadratic (and higher order) 2D and 3D elements, internal nodes may be omitted to reduce the number of degrees of freedom by means of which some higher polynomial terms vanish

"boundary node" elements

⇒ 2D quadratic: 8 nodes term $\xi^2\eta^2$ vanishes

1	ξ	ξ^2
η	$\xi\eta$	$\xi^2\eta$
η^2	$\xi\eta^2$	$\xi^2\eta^2$

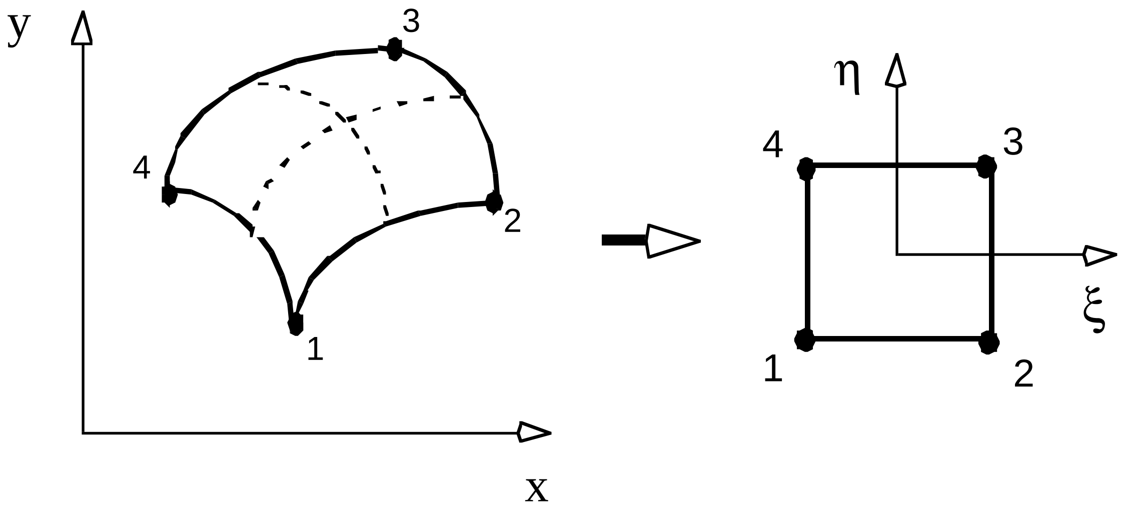
⇒ 3D quadratic: 20 nodes

		include only if node i is defined			
$\varphi_1 =$	$\frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{2}\varphi_5$			$-\frac{1}{2}\varphi_8$
$\varphi_2 =$	$\frac{1}{4}(1+\xi)(1-\eta)$	$-\frac{1}{2}\varphi_5$	$-\frac{1}{2}\varphi_6$		
$\varphi_3 =$	$\frac{1}{4}(1+\xi)(1+\eta)$		$-\frac{1}{2}\varphi_6$	$-\frac{1}{2}\varphi_7$	
$\varphi_4 =$	$\frac{1}{4}(1-\xi)(1+\eta)$			$-\frac{1}{2}\varphi_7$	$-\frac{1}{2}\varphi_8$
$\varphi_5 =$	$\frac{1}{4}(1-\xi^2)(1-\eta)$				
$\varphi_6 =$	$\frac{1}{4}(1+\xi)(1-\eta^2)$				
$\varphi_7 =$	$\frac{1}{4}(1-\xi^2)(1+\eta)$				
$\varphi_8 =$	$\frac{1}{4}(1-\xi)(1-\eta^2)$				

Interpolation functions of four to eight variable-number-nodes for 2D element

General Isoparametric Elements

curvilinear distorted element



$$x = f_x(\xi,\eta,\zeta)\,; \quad y = f_y(\xi,\eta,\zeta)\,; \quad z = f_z(\xi,\eta,\zeta)$$

interpolation by shape functions: $\{\mathrm{x}\} = \{\Psi\}\{\hat{\mathrm{x}}\}_k$

isoparametric elements: $\{\Psi\} = \{\Phi\}$, $\psi_i = \varphi_i$

2D: $\{\mathrm{x}\} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\{\hat{\mathrm{x}}\}_k = \begin{pmatrix} x_1^{(k)} \\ y_1^{(k)} \\ \ldots \\ x_N^{(k)} \\ y_N^{(k)} \end{pmatrix}$

3D: $\{\mathrm{x}\} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\{\hat{\mathrm{x}}\}_k = \begin{pmatrix} x_1^{(k)} \\ y_1^{(k)} \\ z_1^{(k)} \\ \ldots \\ \ldots \end{pmatrix}$

differential operator (2D) $\{D\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_x}{\partial \xi} & \frac{\partial f_y}{\partial \xi} & \frac{\partial f_z}{\partial \xi} \\ \frac{\partial f_x}{\partial \eta} & \frac{\partial f_y}{\partial \eta} & \frac{\partial f_z}{\partial \eta} \\ \frac{\partial f_x}{\partial \zeta} & \frac{\partial f_y}{\partial \zeta} & \frac{\partial f_z}{\partial \zeta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\left\{\frac{\partial}{\partial \xi}\right\} = \{J\}\left\{\frac{\partial}{\partial x}\right\} \qquad \Rightarrow \qquad \left\{\frac{\partial}{\partial x}\right\} = \{J\}^{-1}\left\{\frac{\partial}{\partial \xi}\right\}$$

{J} = JACOBI matrix, $\det\{J\} = |J| \neq 0$

2D: $$\{J\}^{-1} = \frac{1}{|J|}\begin{pmatrix} \frac{\partial f_y}{\partial \eta} & -\frac{\partial f_y}{\partial \xi} \\ -\frac{\partial f_x}{\partial \eta} & \frac{\partial f_x}{\partial \xi} \end{pmatrix}$$

$$|J| = \frac{\partial f_x}{\partial \xi}\frac{\partial f_y}{\partial \eta} - \frac{\partial f_y}{\partial \xi}\frac{\partial f_x}{\partial \eta}$$

$$\frac{\partial f_x}{\partial \xi} = \sum_{i=1}^{N_k} \frac{\partial \varphi_i}{\partial \xi} x_i^{(k)}, \text{ ... etc.}$$

calculation of

strain matrix $\{\varepsilon\} = \{D\}\{\Phi\}\{\hat{u}\}_k = \{B\}\{\hat{u}\}_k$

stress matrix $\{\sigma\} = \{C\}_k\{\varepsilon\} = \{C\}_k\{B\}\{\hat{u}\}_k$

element stiffness matrix

$$\{K\}_k = \iiint_{B_k} \{B\}^T\{C\}_k\{B\}\, dV$$

transformation of the volume differential:

2D: $dV = B\,dA = B\,dx\,dy = B|J|\,d\xi\,d\eta$

(B = thickness)

3D: $dV = dx\,dy\,dz = |J|\,d\xi\,d\eta\,d\zeta$

condition: $|J| > 0$, {J} positive definite

element stiffness matrix

$$\{K\}_k = \int_{\xi=-1}^{+1}\int_{\eta=-1}^{+1}\int_{\zeta=-1}^{+1}\{B\}^T\{C\}_k\{B\}|J|\,d\xi\,d\eta\,d\zeta$$

integration numerically by GAUSS quadrature

global stiffness matrix

$$\{K\} = \sum_{k=1}^{K}\{A\}_k^T\{K\}_k\{A\}_k$$

GAUSS Quadrature

a function $f(\xi)$ is integrated approximately by a

weighted sum of its values at sampling points ξ_j

$$\int_{-1}^{+1} f(\xi)\,d\xi = \sum_{j=1}^{n} w_j\, f(\xi_j)$$

the formula integrates a polynomial of order $(2n\text{-}1)$ exactly

sampling points (GAUSS points):

number n	coordinates ξ_j	weight w_j
1	0.000 000	2.000 000
2	±0.577 350	1.000 000
3	±0.774 597	0.555 556
	0.000 000	0.888 889
4	±0.861 136	0.347 855
	±0.339 981	0.652 145

2D:
$$\int_{\eta=-1}^{+1}\int_{\xi=-1}^{+1} f(\xi)\,d\xi\,d\eta = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k\, f(\xi_j,\eta_k)$$

Nonlinear FE Analyses

nonlinear structural behaviour

a) geometrical (large displacements and/or large strain)
b) constitutive nonlinearity (material)

incremental formulation

quasi-static process (equilibrium),

"time" $t \geq 0$ monotonously increasing parameter of load history

principle of virtual work

$$\text{time } t \quad : \quad \delta A(t) - \delta W(t) = 0$$

$$\text{time } t + \Delta t: \quad \delta A(t+\Delta t) - \delta W(t+\Delta t) = 0$$

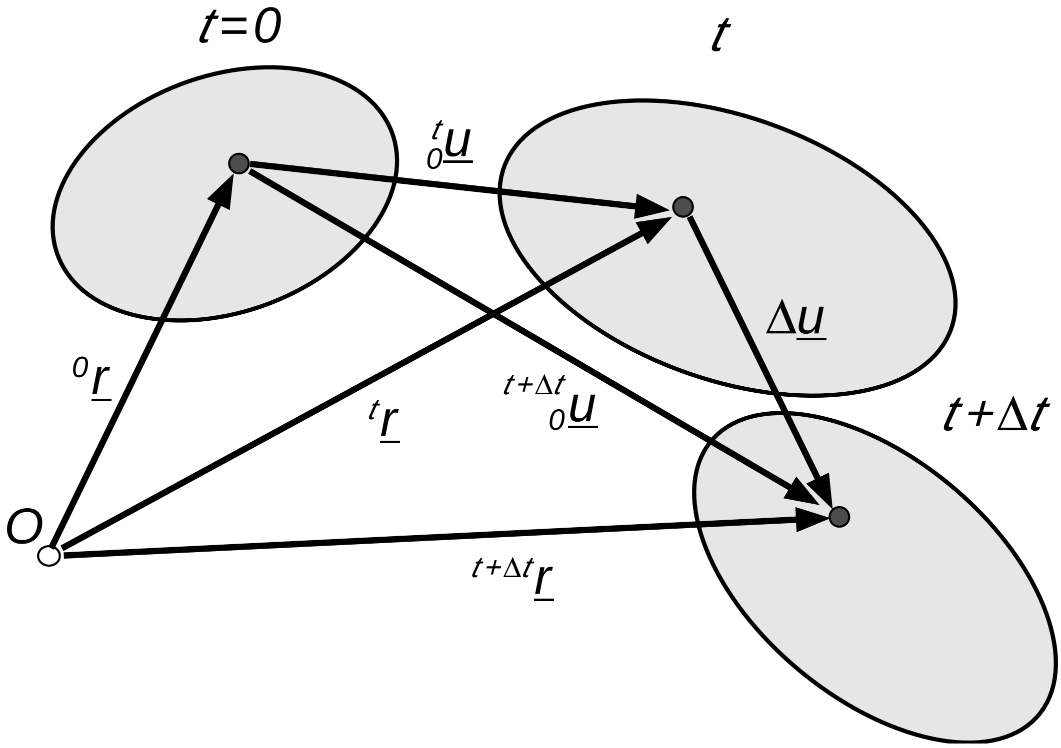
$$f(t+\Delta t) \approx f(t) + \dot{f}\Big|_t \Delta t$$

$$t \to t + \Delta t: \quad \boxed{\delta\Delta A - \delta\Delta W = 0}$$

FE formulation:

$$\boxed{\{\mathrm{K}(\{\mathrm{u}\})\}\,\Delta\{\mathrm{u}\} = \Delta\{\mathrm{f}\}}$$

(a) **geometrical non-linearity**



configuration $\quad {}^{t+\Delta t}\underline{r} = {}^0\underline{r} + {}^{t+\Delta t}_{\ 0}\underline{u} = {}^t\underline{r} + \Delta\underline{u}$

total displacement $\quad {}^{t+\Delta t}_{\ 0}\underline{u} = {}^t_0\underline{u} + \Delta\underline{u}$

total strain (GREEN's quadratic strain)

$$ {}^{t+\Delta t}_{\ 0}\underline{E}^{(G)} = \frac{1}{2}\left[\left(\frac{\partial\, {}^{t+\Delta t}_{\ 0}\underline{u}}{\partial\, {}^0\underline{r}}\right) + \left(\frac{\partial\, {}^{t+\Delta t}_{\ 0}\underline{u}}{\partial\, {}^0\underline{r}}\right)^{T} + \left(\frac{\partial\, {}^t_0\underline{u}}{\partial\, {}^0\underline{r}}\right)\cdot\left(\frac{\partial\, {}^t_0\underline{u}}{\partial\, {}^0\underline{r}}\right)^{T}\right] $$

$$ = {}^{t+\Delta t}_{\ 0}\underline{E} + {}^{t+\Delta t}_{\ 0}\underline{G} $$

left subscript: reference state

left superscript: actual state

reference state *0*: Total LAGRANGEan Formulation

reference state *t:* Updated LAGRANGEan Formulation

Updated LAGRANGEan Formulation

reference state t

$$ {}^{t+\Delta t}_{t}\underline{\underline{S}} = {}^{t}_{t}\underline{\underline{S}} + \Delta_{t}\underline{\underline{S}} = {}^{t}\underline{\underline{T}} + \Delta_{t}\underline{\underline{S}} $$

$\underline{\underline{S}}$ = 2nd PIOLA-KIRCHHOFF stresses

$\underline{\underline{T}}$ = CAUCHY stresses

$$ {}^{t+\Delta t}_{t}\underline{\underline{E}}^{(G)} = \Delta_{t}\underline{\underline{E}}^{(G)} \approx \Delta_{t}\underline{\underline{E}} $$

virtual work during $t \to t + \Delta t$ (surface tractions only):

$$ \delta\Delta A = \int_{\partial^t V} {}^{t+\Delta t}_{t}\underline{t} \cdot \delta\, {}^{t+\Delta t}_{t}\underline{u}\, dA = \int_{\partial^t V} \Delta\underline{t} \cdot \delta\Delta\underline{u}\, dA $$

$$ \delta\Delta W = \int_{{}^t V} {}^{t+\Delta t}_{t}\underline{\underline{S}} \cdot\cdot\, \delta\Delta\, {}^{t+\Delta t}_{t}\underline{\underline{E}}^{(G)}\, dV $$

$$ = \int_{{}^t V} {}^{t}_{t}\underline{\underline{S}} \cdot\cdot\, \delta\Delta_{t}\underline{\underline{E}}\, dV + \int_{{}^t V} \Delta_{t}\underline{\underline{S}} \cdot\cdot\, \delta\Delta_{t}\underline{\underline{E}}\, dV $$

$$ \int_{\partial^t V} \Delta\underline{t} \cdot \delta\, {}^{t}\underline{u}\, dA - \int_{{}^t V} {}^{t}_{t}\underline{\underline{S}} \cdot\cdot\, \delta\Delta_{t}\underline{\underline{E}}\, dV = \delta A(t) - \delta W(t) = 0 $$

$$ \boxed{\int_{{}^t V} \Delta_{t}\underline{\underline{S}} \cdot\cdot\, \delta\Delta_{t}\underline{\underline{E}}\, dV - \int_{\partial^t V} \Delta\underline{t} \cdot \delta\left({}^{t}\underline{u} + \Delta\underline{u}\right) dA = 0} $$

constitutive relation: $\Delta_{t}\underline{\underline{S}} = {}_{t}\underline{\underline{\underline{\underline{C}}}} \cdot\cdot\, \Delta_{t}\underline{\underline{E}}$

(incrementally linear)

$$ \boxed{{}_{t}\{K(\Delta\{u\})\}\,\Delta\{u\} = \Delta\{f\}} $$

nonlinear system of equations

$\Rightarrow$ iterative solution (e.g. NEWTON)

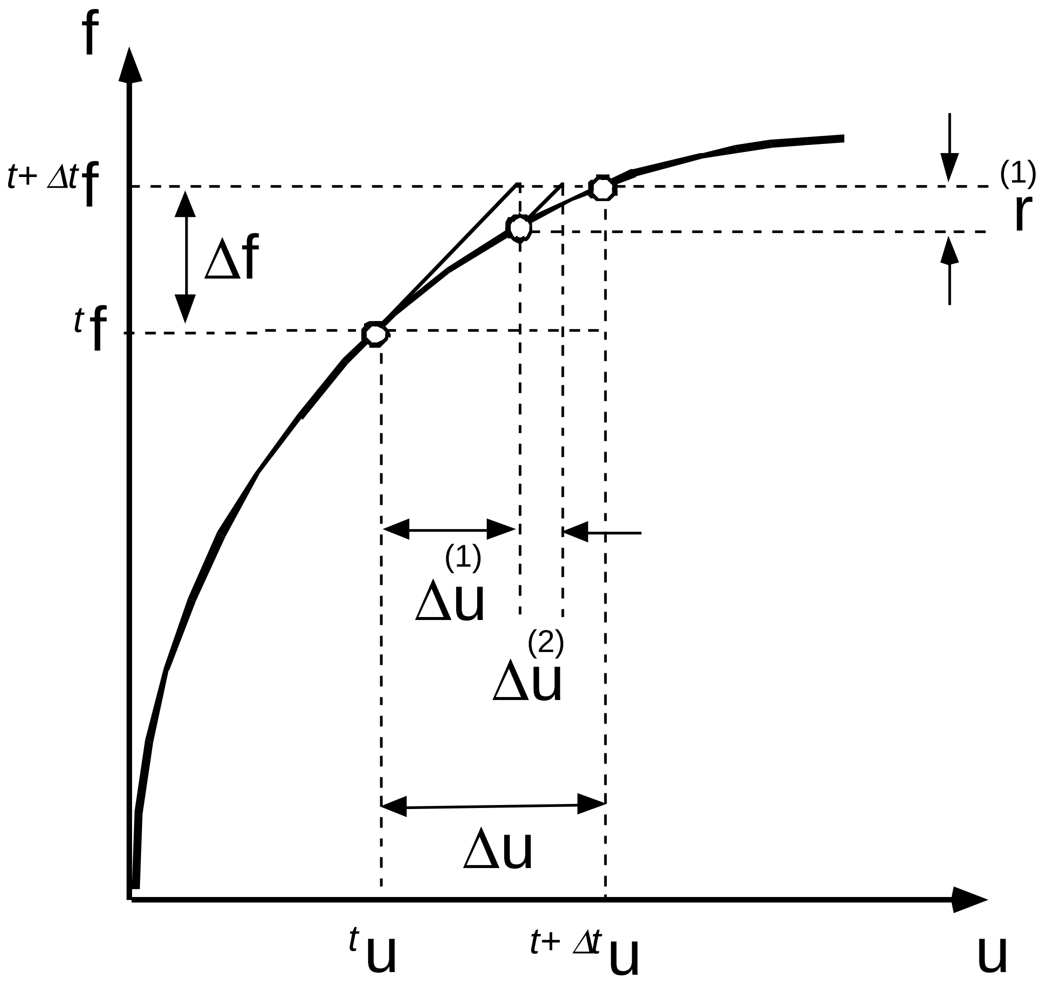
$$\Delta\{u\} = \sum_{j=1}^{i} \Delta\overset{(j)}{\{u\}} = \sum_{j=1}^{i} {}_{t}\{K\}^{-1}\overset{(j)}{\{r\}}$$

residual force (out-of balance force)

$$\overset{(j)}{\{r\}} = {}^{t+\Delta t}\{f\} - \{f({}^{t}\{u\} + \Delta\overset{(j)}{\{u\}})\}$$

convergence criterion

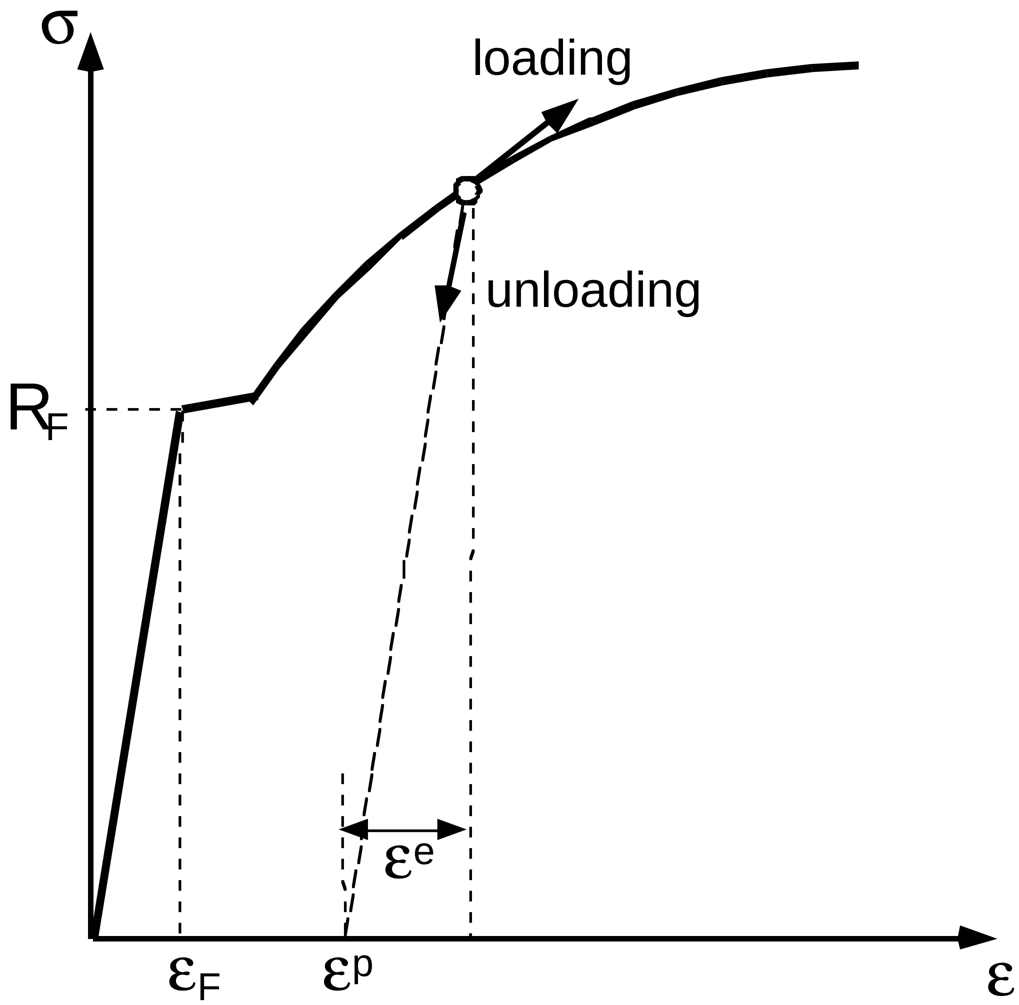
$$\left\| \overset{(i)}{\{r\}} \right\| \leq \text{RTOL} \quad \text{and / or} \quad \left\| \Delta\overset{(i)}{\{u\}} \right\| \leq \text{DTOL}$$



(b) **constitutive non-linearity**

elastic- plastic material behaviour
(VON MISES, PRANDTL, REUSS)

1. uniaxial tensile test



yield condtion: $\sigma \leq R(\varepsilon^p) \quad , \quad R(0) = R_\mathrm{F}$

HOOKE's law: $\sigma = \begin{cases} E\varepsilon & \text{for} \quad \sigma \leq R_\mathrm{F} \\ E\left(\varepsilon - \varepsilon^\mathrm{p}\right) & \text{for} \quad \sigma > R_\mathrm{F} \end{cases}$

loading condtion: $\begin{cases} \dot{\sigma} > 0, \quad \dot{\varepsilon}^p > 0 & \text{loading} \\ \dot{\sigma} < 0, \quad \dot{\varepsilon}^p = 0 & \text{unloading} \end{cases}$

2. general (multiaxial) stress state

yield condtion: $\bar{\sigma} \le R(\varepsilon^p)$; $\bar{\sigma} = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}}$

VON MISES equivalent (effective) stress

$\sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}$ deviatoric stresses

HOOKE's law:

$$\dot{\sigma}_{ij} = \begin{cases} 2G\left[\dot{\varepsilon}_{ij} + \dfrac{\nu}{1-2\nu}(\varepsilon_{kk})\,\delta_{ij}\right] & \text{for} \quad \bar{\sigma} \le R_\mathrm{F} \\ 2G\left[\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^p_{ij}\right) + \dfrac{\nu}{1-2\nu}(\varepsilon_{kk})\,\delta_{ij}\right] & \text{for} \quad \bar{\sigma} > R_\mathrm{F} \end{cases}$$

loading condtion:
$$\begin{cases} \sigma'_{ij}\dot{\sigma}_{ij} > 0, \quad \dot{\varepsilon}^p_{ij} > 0 & \text{loading} \\ \sigma'_{ij}\dot{\sigma}_{ij} < 0, \quad \dot{\varepsilon}^p_{ij} = 0 & \text{unloading} \end{cases}$$

flow rule
$$\dot{\varepsilon}^p_{ij} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}^p}{\bar{\sigma}}\sigma'_{ij} = \frac{3}{2}\frac{\dot{\bar{\sigma}}}{E^p\bar{\sigma}}\sigma'_{ij}$$

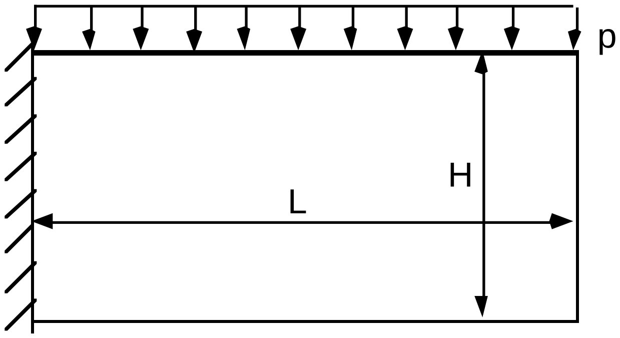
$$E^p = \frac{E_\mathrm{t}E}{E - E_\mathrm{t}} \quad , E_\mathrm{t} = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}$$

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = \bar{\sigma}\dot{\bar{\varepsilon}}^p \quad \Rightarrow \dot{\bar{\varepsilon}}^p = \sqrt{\tfrac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}}$$

$\dot{\varepsilon}^p_{ij}$ is deviatoric, i.e. $\dot{\varepsilon}^p_{kk} = 0$

"true" stress-strain curve required for UL formulation:

true stresses $\sigma = \dfrac{F}{A}$ vs logarithmic strain $\varepsilon = \ln\dfrac{L}{L_0}$



A plane panel of dimensions

length L = 200 mm, height H = 100 mm, thickness B = 1mm

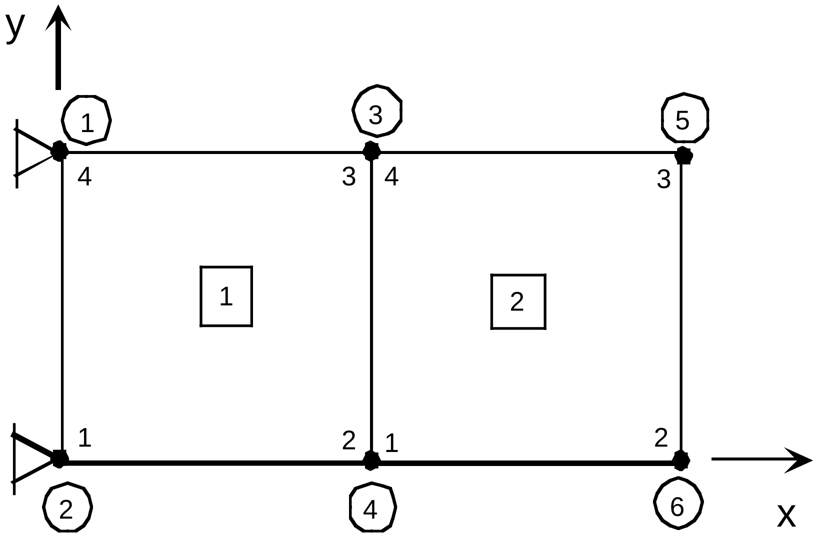
is clamped at x = 0 and loaded by a constant pressure p = 100 MPa at y = H

The material is isotropic, elastic with

YOUNG's modulus E = 218 400 MPa and POISSON's ratio ν = 0.3.

Establish the system of equations of the respective finite element model and calculate the displacement, stress and strain field by applying the FE code ANSYS.

1. analytical solution for a model of two linear elements



1.1 Calculate the elastic stiffness matrix

$$\{\mathrm{C}\} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

for plane stress conditions.

1.2 Calculate the (symmetric) element stiffness matrix for one element k

$$\{\mathrm{K}\}_k = \frac{a_k b_k}{4} \int_{\xi=-1}^{+1} \int_{\eta=-1}^{+1} \{\mathrm{B}\}^T \{\mathrm{C}\}_k \{\mathrm{B}\} d\xi\, d\eta$$

1.3 Calculate the global stiffness matrix (see table on page 3)

$$\{\mathrm{K}\} = \sum_{k=1}^{K} \{\mathrm{A}\}_k^T \{\mathrm{K}\}_k \{\mathrm{A}\}_k$$

1.4 Calculate the column matrix of nodal forces (see table on page 3)

$$\{\mathrm{f}\} = \sum_{k=1}^{K} \{\mathrm{A}\}_k^T \int_{\partial S_k} \{\Phi\}^T \{\mathrm{t}\} ds \; .$$

2. FE solution by the ANSYS code

Solve the same problem by the FE code ANSYS and compare the results with analytical values (see lecture notes), especially the displacement u_y of node (6) and the stresses at the nodes (1) and (2) in the table on page 4.

2.1 Use the model of two linear plane stress elements as on page 1.

2.2 Use two quadratic (8-node) elements instead of linear elements.

2.3 Use two other refined meshes and quadratic elements; explain and display the meshes.

analytical solution (theory of bending)

bending stress

$$\sigma_{xx}(x,y) = -\frac{M_z(x)}{I_z}\left(y - \frac{H}{2}\right)$$

$$M_z(x) = -\frac{pBL^2}{2}\left[1 - 2\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right]$$

$$\sigma_{xx}^{\max} = \sigma_{xx}\Big|_{\substack{x=0\\y=H}} = \left|\sigma_{xx}\Big|_{\substack{x=0\\y=0}}\right| = \frac{|M_z(0)|}{W_z} = 1200\,\mathrm{MPa}$$

section modulus $W_z = I_z\,\frac{2}{H} = \frac{BH^2}{6} = \frac{10^4}{6}\,\mathrm{mm}^3$

deflection

1. simple bending (no shear)

$$u_y^b(x) = -\frac{pBL^4}{24EI_z}\left[6\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]$$

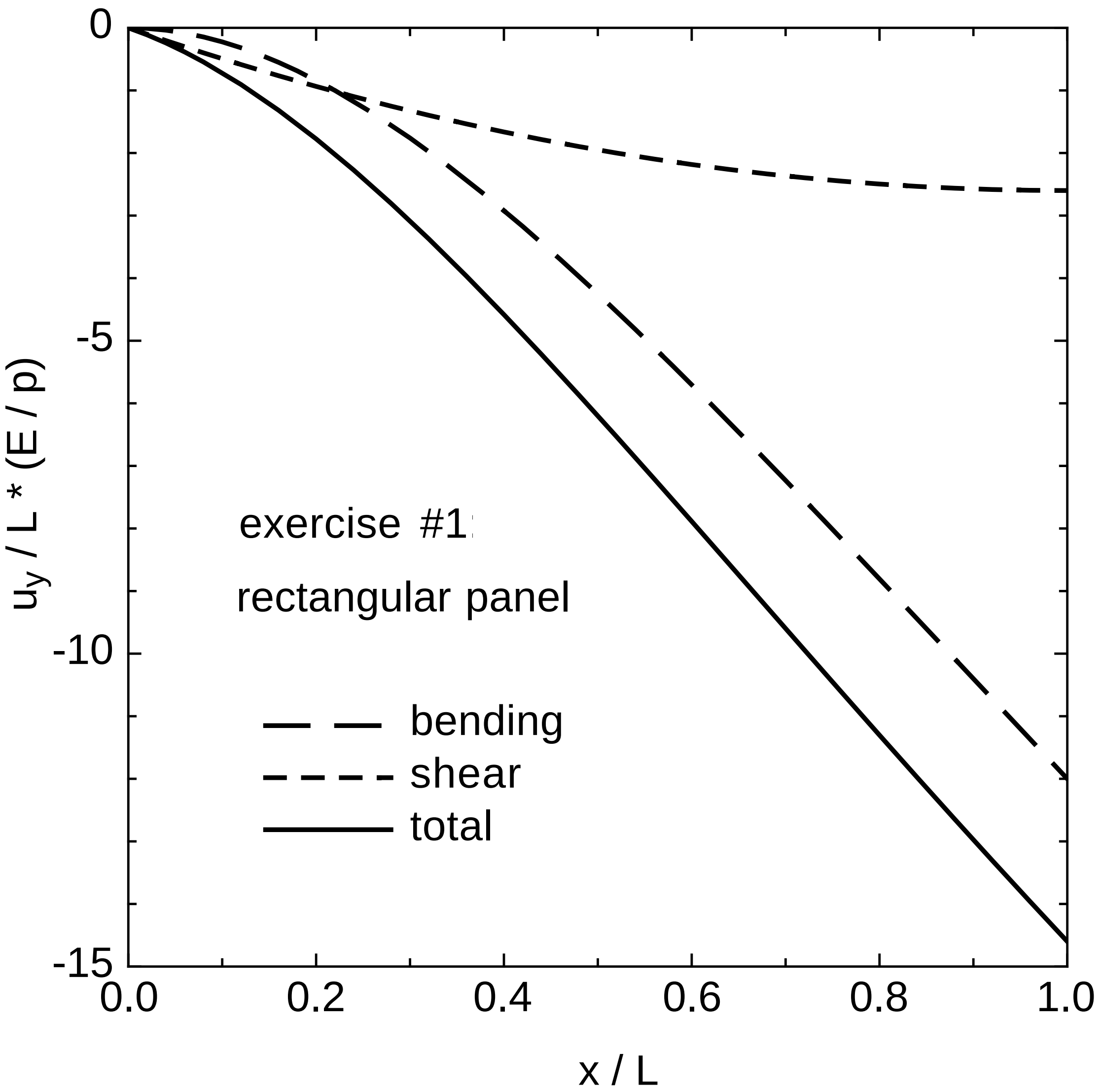
$$u_y^{b,\max} = |u_y(L)| = \frac{pBL^4}{8EI_z} = 1.10\ \mathrm{mm}$$

2. shear

$$u_y^s(x) = -\frac{pL^2}{2GH}\left[2\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right]$$

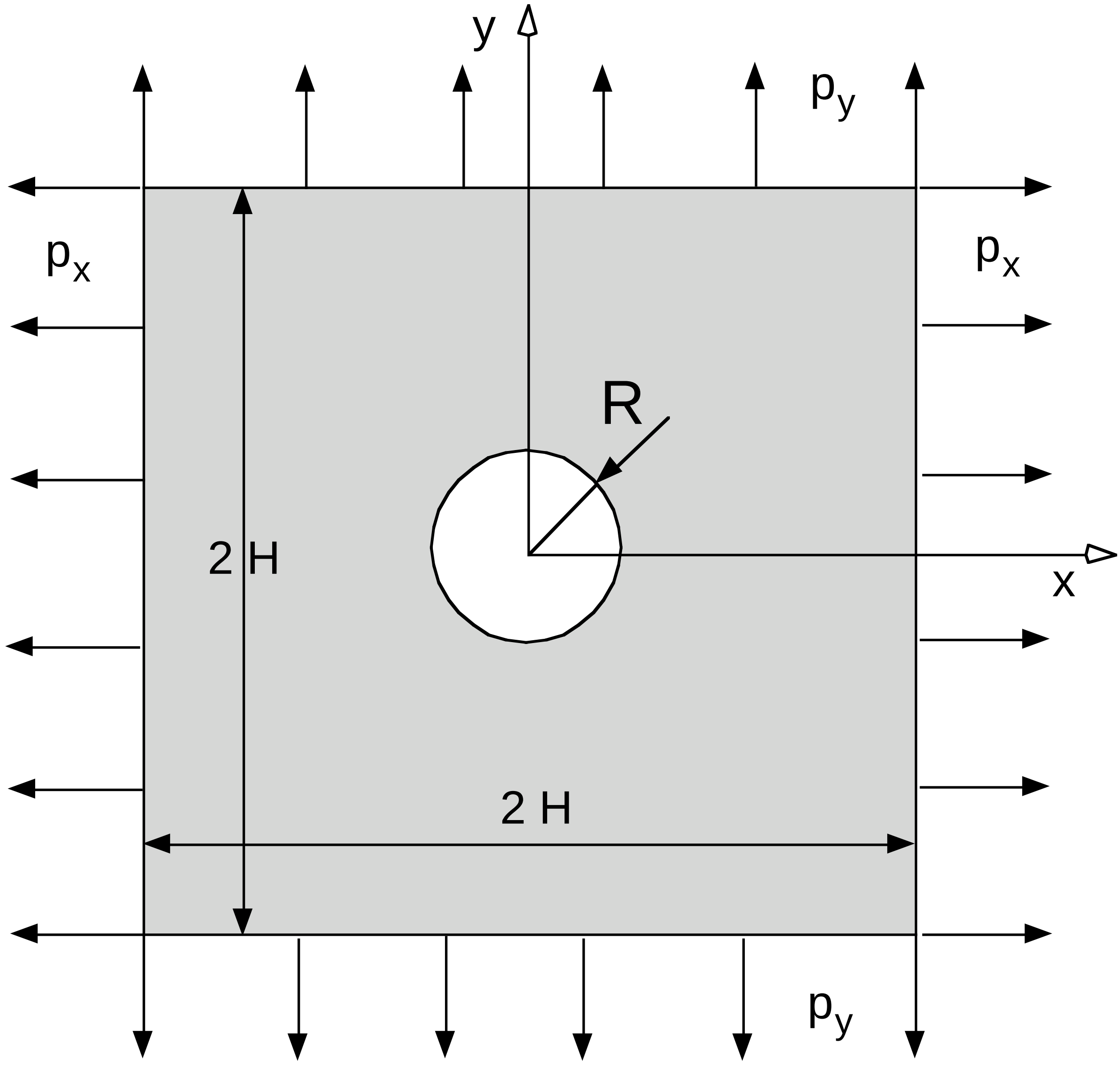
$$u_y^{s,\max} = |u_y^s(L)| = \frac{pL^2}{2GH} = 0.24\ \mathrm{mm}$$

total $\quad u_y^{\max} = u_y^{b,\max} + u_y^{s,\max} = 1.34\ \mathrm{mm}$


0
-5
-10
-15
0.0
0.2
0.4
0.6
0.8
1.0
x / L
u_y / L * (E / p)
exercise #1
rectangular panel
bending
shear
total

exercise #2:

biaxially loaded panel with circular hole



H = 50 mm, R = 10 mm

plane stress, B = 1 mm

p_y = 100 MPa, $p_x = \beta\ p_y$

E = 200000MPa, ν = 0.3

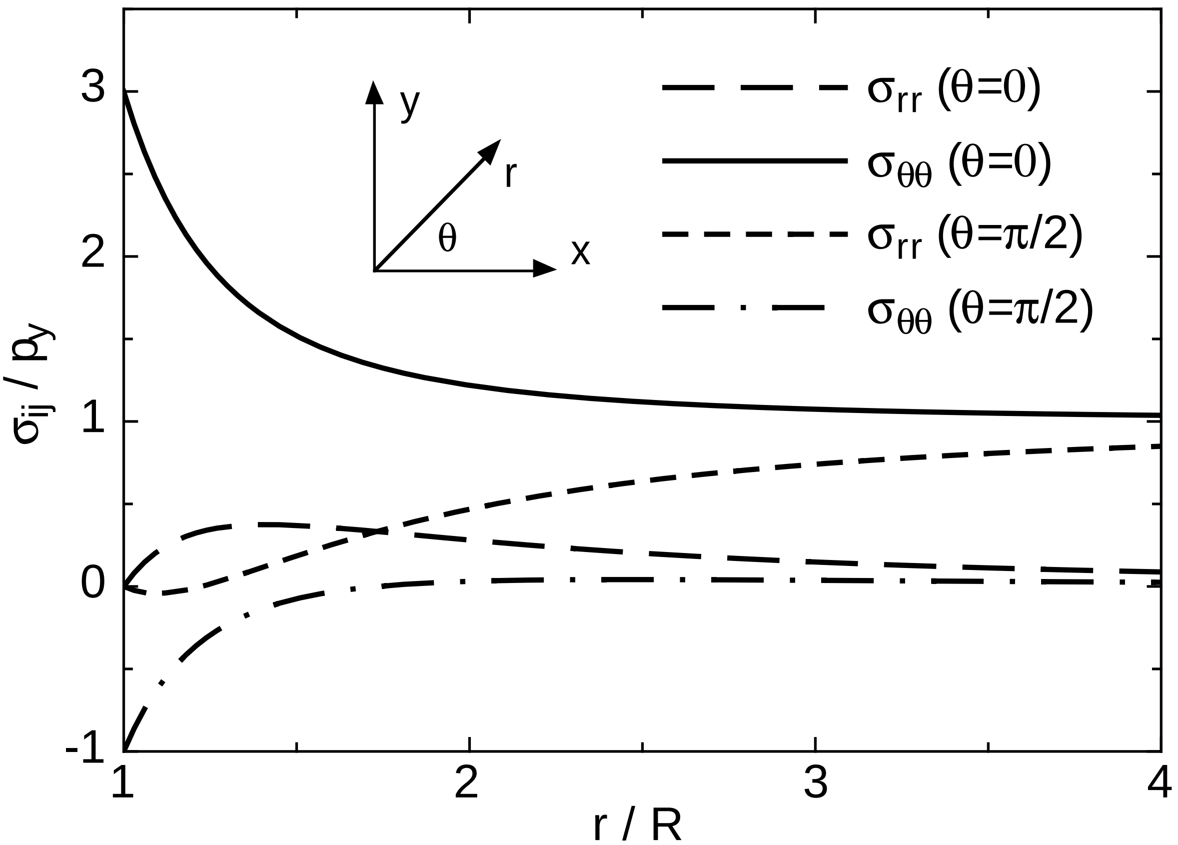
analytical solution for "infinite" plate: *R* << *H*,

uniaxial tension ($\beta = 0$)

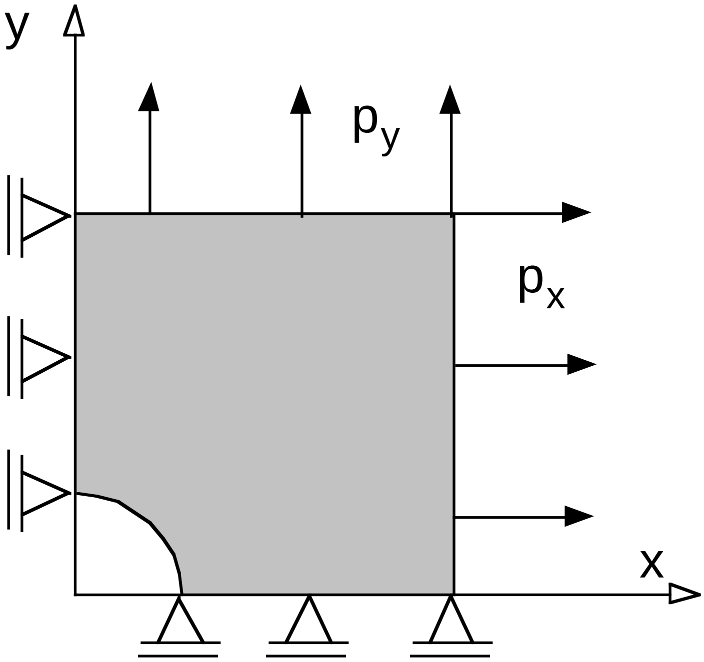
$$\sigma_{rr}(r,\theta) = \tfrac{1}{2}\, p_y \left\{ 1 - \left(\frac{R}{r}\right)^2 - \left[1 - 4\left(\frac{R}{r}\right)^2 + 3\left(\frac{R}{r}\right)^4 \cos 2\theta \right] \right\}$$

$$\sigma_{\theta\theta}(r,\theta) = \tfrac{1}{2}\, p_y \left\{ 1 + \left(\frac{R}{r}\right)^2 + \left[1 + 3\left(\frac{R}{r}\right)^4 \right] \cos 2\theta \right\}$$

$$\sigma_{r\theta}(r,\theta) = \tfrac{1}{2}\, p_y \left[1 + 3\left(\frac{R}{r}\right)^2 - 3\left(\frac{R}{r}\right)^4 \right] \sin 2\theta$$



FE solution: symmetry conditions ⇒1/4 model



boundary conditions:

$$u_y\big|_{y=0} = 0 \quad ; \quad u_x\big|_{x=0} = 0$$

Notation

30.10.99

tensors, vektors, skalars - general

scalar		latin or greek, small or capital (italics)	a , H, α ,
vector			
	symbolic	latin or greek, small,	
		bold or underlined	$\mathbf{x}$, $\underline{x}$, $\boldsymbol{\sigma}_n$, $\underline{\sigma}_n$
	indexed	latin or greek, small (italics)	x_i , α_i
			$\mathbf{x} = x_i\, \mathbf{e}_i$, $\underline{x} = x_i\, \underline{e}_i$
2nd order tensor			
	symbolic	latin or greek, capital,	
		bold or underlined	$\mathbf{T}$, $\underline{T}$
	indexed	latin or greek, capital or small (italics)	T_{ij} , σ_{ij}
			$\mathbf{T} = T_{ij}\, \mathbf{e}_i\, \mathbf{e}_j$, $\underline{T} = T_{ij}\, \underline{e}_i\, \underline{e}_j$
4th order tensor			
	symbolic	latin or greek, capital,	
		bold with overhead <4>	$\overset{<4>}{\mathbf{C}}$
		or double underlined	$\underline{\underline{C}}$
	indexed	latin or greek, capital	C_{ijkl}

$$\overset{<4>}{\mathbf{C}} = C_{ijkl}\, \mathbf{e}_i\, \mathbf{e}_j\, \mathbf{e}_k\, \mathbf{e}_l\ ,\ \underline{\underline{C}} = C_{ijkl}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k\, \underline{e}_l$$

vector and tensor algebra and analysis

scalar product

$\alpha = \mathbf{u}\cdot\mathbf{v} = u_i\, v_i$, $\mathbf{t} = \mathbf{n}\cdot\mathbf{S} = n_i \sigma_{ij}\, \mathbf{e}_j$,

$\mathbf{A} = \mathbf{B}\cdot\mathbf{C} = B_{ij}\, C_{jk}\, \mathbf{e}_i\, \mathbf{e}_k = A_{ik} \mathbf{e}_i\, \mathbf{e}_k$

double scalar product

$\alpha = \mathbf{B}\cdot\cdot\mathbf{C} = B_{ij}\, C_{ji}$ or $\alpha = \mathbf{B} : \mathbf{C}$ ...

$\mathbf{T} = \overset{<4>}{\mathbf{C}} \cdot\cdot \mathbf{E} = C_{ijkl}\, E_{kl}\, \mathbf{e}_i\, \mathbf{e}_j$ or $\underline{S} = \underline{\underline{C}} : \underline{E}$ ...

tensorial product

$\mathbf{T} = \mathbf{a}\, \mathbf{b} = a_i\, b_j\, \mathbf{e}_i\, \mathbf{e}_j$

$\overset{<4>}{\mathbf{C}} = \mathbf{A}\, \mathbf{B} = A_{ij}\, B_{kl}\, \mathbf{e}_i\, \mathbf{e}_j\, \mathbf{e}_k\, \mathbf{e}_l$

$\mathbf{a}\, \mathbf{b}\cdot\mathbf{c} = (\mathbf{a}\, \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\, (\mathbf{b}\cdot\mathbf{c})$

spatial derivatives

NABLA operator $\nabla = \mathbf{e}_i \dfrac{\partial}{\partial x_i}$

$$\mathbf{v} = \text{grad}\, \varphi = \boldsymbol{\nabla}\, \varphi = \frac{\partial \varphi}{\partial x_i}\, \mathbf{e}_i = \varphi_{,i}\, \mathbf{e}_i$$

$$\mathbf{F} = \text{grad}\, \mathbf{v} = \boldsymbol{\nabla}\, \mathbf{v} = \frac{\partial v_i}{\partial x_j}\, \mathbf{e}_i\, \mathbf{e}_j = v_{i,j}\, \mathbf{e}_i\, \mathbf{e}_j$$

$$a = \text{div}\, \mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} = = v_{i,i} = \frac{\partial v_i}{\partial x_i}$$

$$\mathbf{w} = \text{div}\, \mathbf{T} = \boldsymbol{\nabla}\cdot\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\, \mathbf{e}_j = T_{ij,i}\, \mathbf{e}_j$$

convention of summation is used in all cases

$$A_{ij} B_{jk} = \sum_{j=1}^{3} A_{ij} B_{jk} \quad , \quad C_{kk} = \sum_{k=1}^{3} C_{kk} \quad , \quad A_{ij} B_{ji} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ji}$$

special tensors and invariants

2nd order unit tensor $\mathbf{I} = \delta_{ij}\, \mathbf{e}_i\, \mathbf{e}_j = \mathbf{e}_i\, \mathbf{e}_i$

$\mathbf{T}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{T} = \mathbf{T}$

deviator of a tensor **T**: $\mathbf{T}' = T'_{ij}\, \mathbf{e}_i\, \mathbf{e}_j = (T_{ij} - T_{kk}\, \delta_{ij})\, \mathbf{e}_i\, \mathbf{e}_j = \mathbf{T} - \mathbf{T}\cdot\cdot\mathbf{I}$

transposed tensor $\mathbf{T}^{T} = T_{ji}\, \mathbf{e}_i\, \mathbf{e}_j$

inverse tensor $\mathbf{T}^{-1}$, $\underline{T}^{-1}$

$\mathbf{T}\cdot\mathbf{T}^{-1} = \mathbf{T}^{-1}\cdot\mathbf{T} = \mathbf{I}$

1st invariant (trace) of a tensor $J_1(\mathbf{T}) = \mathrm{tr}\,(\mathbf{T})$

2nd invariant of a tensor $J_2(\mathbf{T}) = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{T}^2 - \mathrm{tr}^2\,\mathbf{T}\right)$

3rd invariant (determinant) of a tensor $J_3(\mathbf{T}) = \det\,(\mathbf{T})$

tensors used in continuum mechanics

deformation gradient $\mathbf{F} = \mathbf{I} + \text{grad}\,\mathbf{u} = (\delta_{ij} + u_{i,j})\,\mathbf{e}_i\,\mathbf{e}_j$

GREEN's strain tensor $\mathbf{E}^{(G)} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{i,k}u_{k,j}\right)\mathbf{e}_i\mathbf{e}_j$

linear strain tensor $\mathbf{E} = \frac{1}{2}\left(\text{grad}\,\mathbf{u} - \text{grad}^{\mathrm{T}}\,\mathbf{u}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\mathbf{e}_i\mathbf{e}_j = \varepsilon_{ij}\,\mathbf{e}_i\mathbf{e}_j = \mathbf{E}^T$

strain rate tensor $\mathbf{D} = d_{ij}\,\mathbf{e}_i\mathbf{e}_j = \frac{1}{2}\left(\text{grad}\,\mathbf{v} - \text{grad}^{\mathrm{T}}\,\mathbf{v}\right)$

for small strains $\mathbf{D} = \dot{\mathbf{E}} = \dot{\varepsilon}_{ij}\,\mathbf{e}_i\mathbf{e}_j$

elastic and plastic part $\dot{\mathbf{E}} = \dot{\mathbf{E}}^{\mathrm{e}} + \dot{\mathbf{E}}^{\mathrm{p}} = \left(\dot{\varepsilon}_{ij}^{\mathrm{e}} + \dot{\varepsilon}_{ij}^{\mathrm{p}}\right)\mathbf{e}_i\mathbf{e}_j$

accumulated effective ("equivalent") plastic strain

$$\varepsilon_{\mathrm{e}}^{\mathrm{p}} = \bar{\varepsilon}^{p}\int_0^t \sqrt{\tfrac{2}{3}\dot{\varepsilon}_{ij}^{\mathrm{p}}\dot{\varepsilon}_{ij}^{\mathrm{p}}}\,d\tau = \int_0^t \sqrt{\tfrac{4}{3}J_2(\dot{\mathbf{E}}^{\mathrm{p}})}\,d\tau$$

CAUCHY ("true") stress tensor $\mathbf{S} = \sigma_{ij}\,\mathbf{e}_i\,\mathbf{e}_j = \mathbf{S}^{\mathrm{T}} = \sigma_{ji}\,\mathbf{e}_i\,\mathbf{e}_j$

hydrostatic stress $\sigma_{\mathrm{h}} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\,\mathrm{tr}\,\mathbf{S} = \frac{1}{3}J_1(\mathbf{S})$

VON MISES effective ("equivalent") stress

$$\sigma_{\mathrm{e}} = \bar{\sigma} = \sqrt{3J_2(\mathbf{S}')} = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

2nd PIOLA KIRCHHOFF stress tensor $\mathbf{T} = \det(\mathbf{F})\,\mathbf{F}^{-1} \cdot \mathbf{S} \cdot \mathbf{F}^{-T} = \mathbf{T}^{\mathrm{T}}$

for small strains $\mathbf{T} = \mathbf{S}$

plastic yielding

yield curve (uniaxial tensile test)	$R(\varepsilon_\mathrm{p})$
with	$\varepsilon_\mathrm{p} = \varepsilon_\mathrm{e}^\mathrm{p} = \varepsilon - \dfrac{\sigma}{E}$
yield strength (at initial plastic flow)	$R_0 = R(0)$
especially	
lower yield strength	R_eL
0.2% proof stress	$R_\mathrm{p0.2}$

plastic potential, flow potential (VON MISES)

isotropic hardening $\quad \Phi = \sigma_\mathrm{e}^2 - R^2(\varepsilon_\mathrm{p}) = \frac{3}{2}\sigma'_{ij}\sigma'_{ij} - R^2(\varepsilon_\mathrm{p}) = \mathbf{S}' \cdot\cdot \mathbf{S}' - R^2(\varepsilon_\mathrm{p})$

kinematic hardening $\quad \Phi = \sigma_\mathrm{e}^2 - R_0^2 = \frac{3}{2}(\mathbf{S}' - \mathbf{X}')\cdot\cdot(\mathbf{S}' - \mathbf{X}') - R_0^2$

with $\mathbf{X}$ = "back stress" tensor

associated flow rule

$$\dot{\varepsilon}_{ij}^\mathrm{p} = \dot{\lambda}\frac{\partial \Phi}{\partial \sigma'_{ij}} \quad , \quad \dot{\mathbf{E}}^p = \dot{\lambda}\frac{\partial \Phi}{\partial \mathbf{S}'}$$

matrix notation

general: $(n\times m)$ matrix: {A} or A

n = number of rows

m = number of columns

$$\{A\} = \begin{pmatrix} A_{11} & . & . & . & A_{1m} \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{n1} & . & . & . & A_{nm} \end{pmatrix} \qquad \{A\}^T = \begin{pmatrix} A_{11} & . & . & A_{n1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ A_{1m} & . & . & A_{nm} \end{pmatrix}$$

special: $(n\times 1)$ matrix {u} or u

$$\{u\} = \begin{pmatrix} u_1 \\ . \\ u_n \end{pmatrix} \qquad \{u\}^T = \begin{pmatrix} u_1 & . & u_n \end{pmatrix}$$

the elements of a matrix **do not** form the components of a tensor or vector, in general

products

$$\{u\}^T\{v\} = \alpha = \sum_1^n u_i v_i$$ {u} and {v}: $(n\times 1)$; α scalar

$$\{u\}\{v\}^T = \{A\} = \begin{pmatrix} u_1v_1 & . & . & . & u_1v_m \\ . & . & . & . & . \\ . & . & . & . & . \\ u_nv_1 & . & . & . & u_nv_m \end{pmatrix}$$ {u}: $(n\times 1)$, {v}: $(m\times 1)$; {A}: $(n\times m)$

$$\{v\} = \{A\}\{u\} = \begin{pmatrix} \sum_{i=1}^m A_{1i}v_i \\ .... \\ \sum_{i=1}^m A_{ni}v_i \end{pmatrix}$$ {A}: $(n\times m)$; {u}: $(m\times 1)$, {v}: $(n\times 1)$

$$\{v\}^T = \{u\}^T\{A\}^T$$ $\{A\}^T$: $(m\times n)$; $\{u\}^T$: $(1\times m)$, $\{v\}^T$: $(1\times n)$

$$\{C\} = \{A\}\{B\} = \begin{pmatrix} \sum_{i=1}^m A_{1i}B_{i1} & . & . & \sum_{i=1}^m A_{1i}B_{im} \\ .... & . & . & .... \\ \sum_{i=1}^m A_{ni}B_{i1} & . & . & \sum_{i=1}^m A_{ni}B_{im} \end{pmatrix}$$

{A}: $(n\times m)$; {B}: $(m\times p)$; {C}: $(n\times p)$

Physical Units (SI-Units)

ISO 1000 (1973)

SI = Système International d'Unités

SI units are
the seven **basic units** and coherently **derived units**, i.e. by a factor of 1

Basic Quantities and Units

basic quantity	SI basic unit	
	name	symbol
length	meter	m
mass	kilogramm	kg
time	second	s
thermodyn. temperature	Kelvin	K
amperage, intensity of electric current	Ampere	A
amount of substance	Mol	mol
luminous intensity	Candela	cd

Decimal Parts and Multiples of SI Units

Parts and multiples of SI units which are generated by multiplication with the factors

$$10^{\pm 1},\ 10^{\pm 2},\ 10^{\pm 3n}\ (n = 1, 2, ...)$$

have special names and symbols.
They are composed by prefixes.

faktor	prefix	symbol
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	µ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deca	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T

Multiples by the factors

$$10^{\pm 3n}\ (n = 1, 2, ...)$$

are to be preferred!

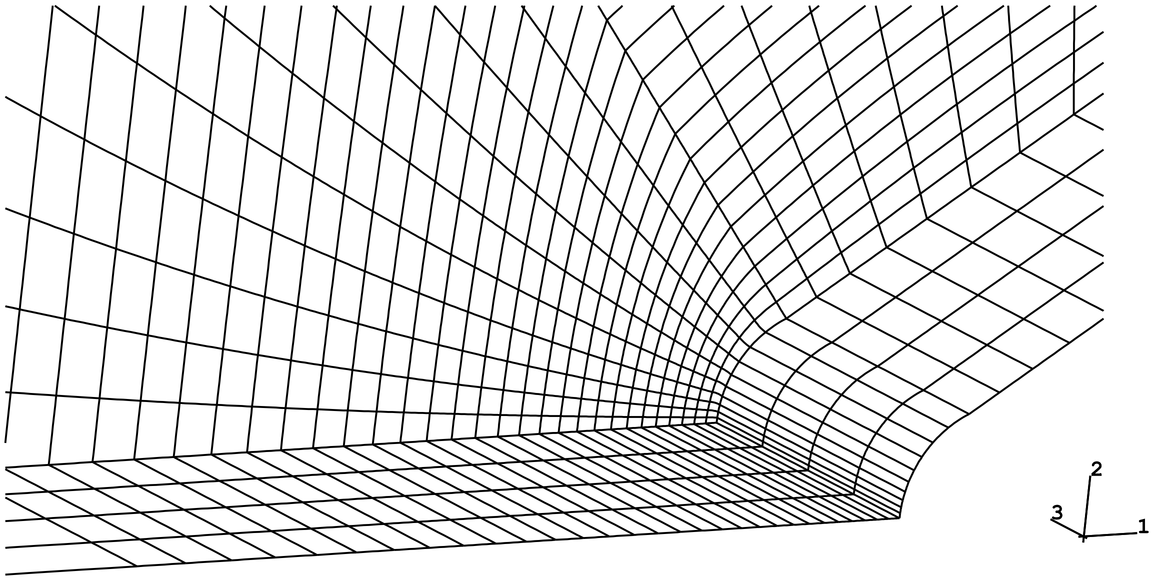
Derived Quantities and Units

Derived units are formed by products or ratios of basic units.
The same holds for the unit symbols.

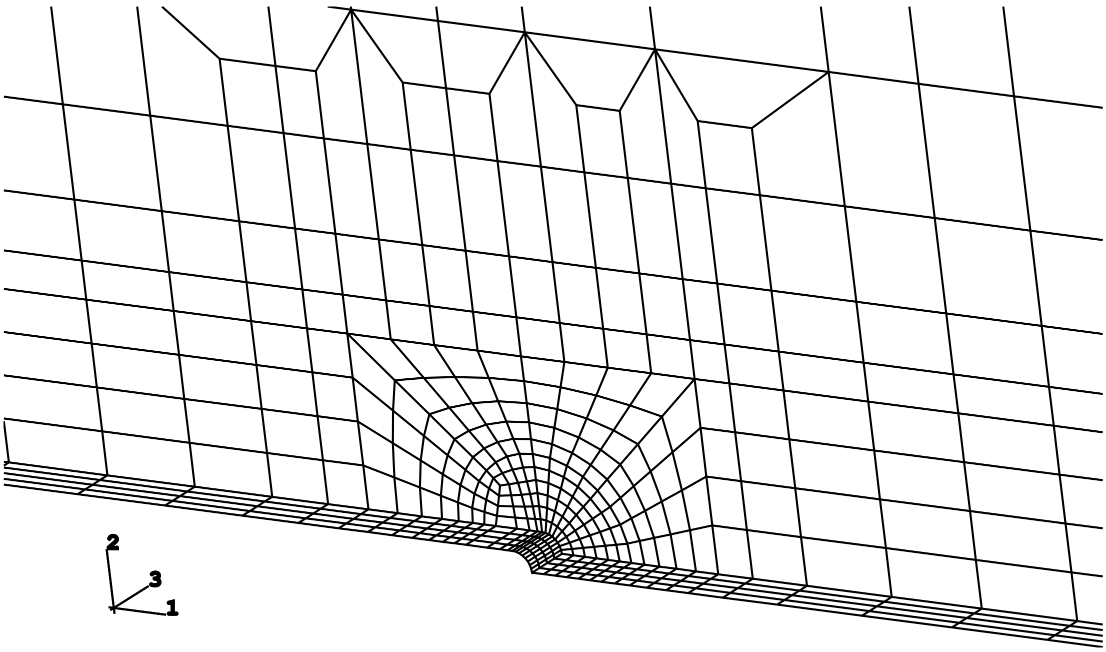
quantity	SI unit		relation
	name	symbol	
frequency	Hertz	Hz	1 Hz = 1 s^{-1}
force	Newton	N	1 N = 1 kg m s^{-2}
pressure, stress	Pascal	Pa	1 Pa = 1 N / m^2
energy, work, amount of heat	Joule	J	1 J = 1 N m
power, heat flow	Watt	W	1 W = 1 J / s
electric charge, quantity of electricity	Coulomb	C	1 C = 1 A s
electric potential, voltage	Volt	V	1 V = 1 J / C
electric capacity	Farad	F	1 F = 1 C / V
electric resistance	Ohm	Ω	1 Ω = 1 V / A
magnetic flux	Weber	Wb	1 Wb = 1 V s
magnetic flux density	Tesla	T	1 T = 1 Wb / m^2
inductivity	Henry	H	1 H = 1 Wb / A

Examples for FE meshes

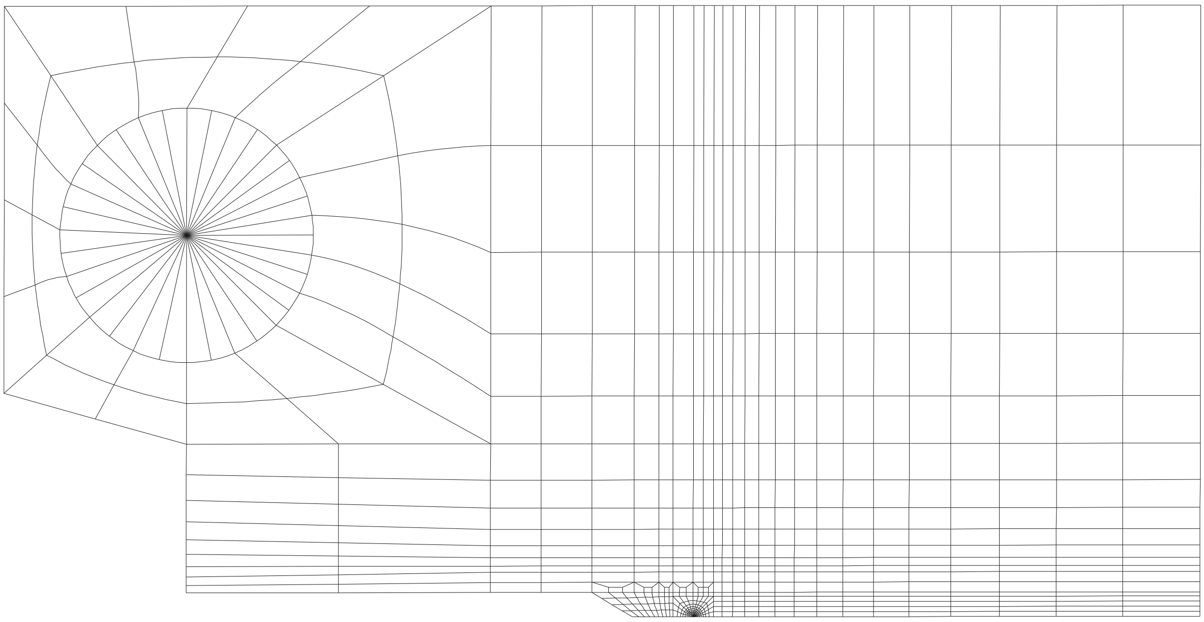
3D FE mesh at the notch of a side-notched flat tensile panel



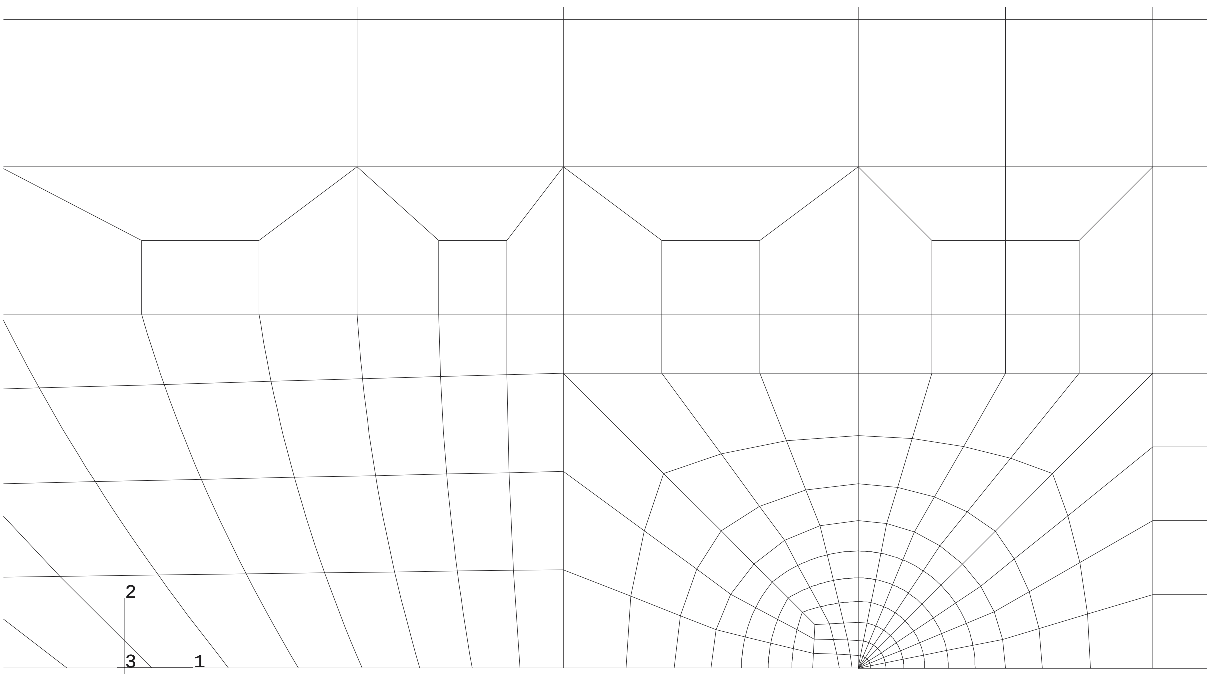
3D FE mesh of a centre-notched tensile panel



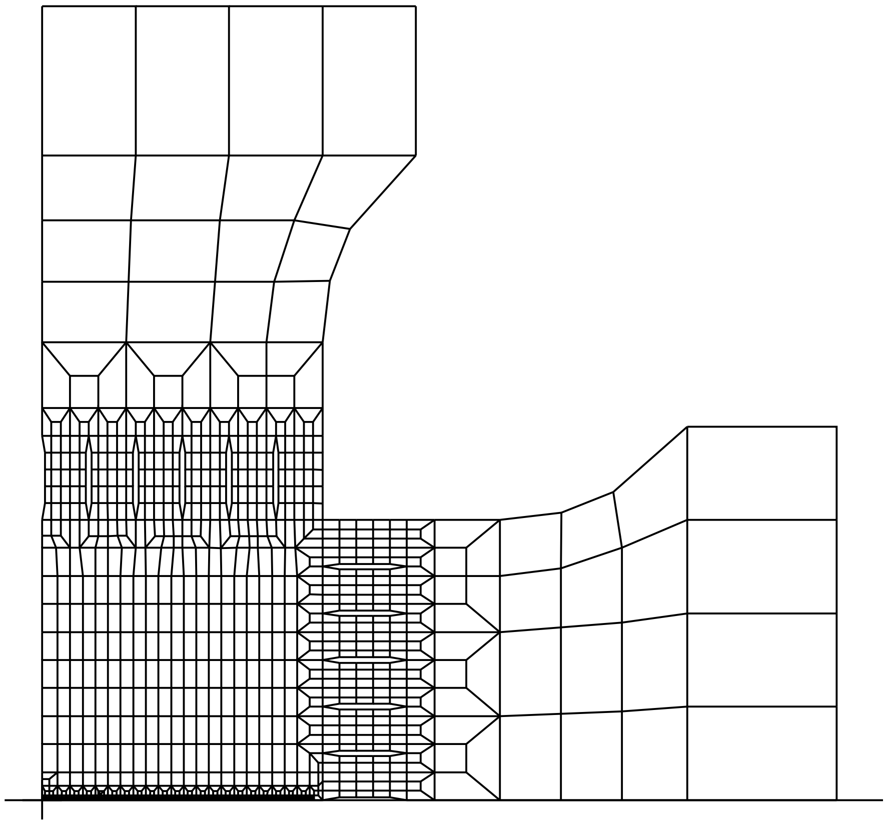
2D FE mesh of a C(T) specimen, half model accounting for symmetry



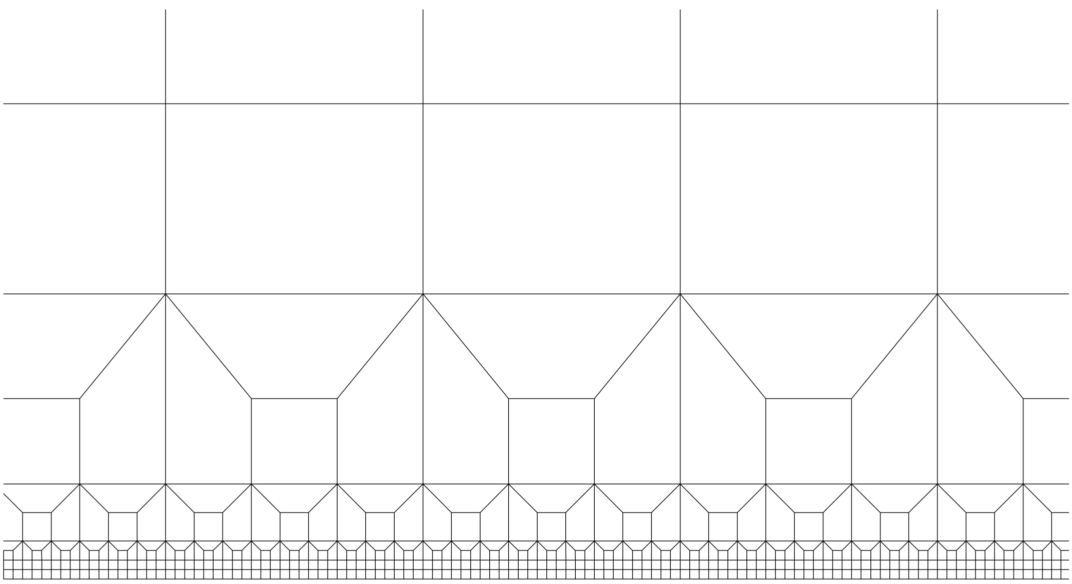
Detail of the FE mesh of the C(T) specimen at the crack tip with collapsed elements



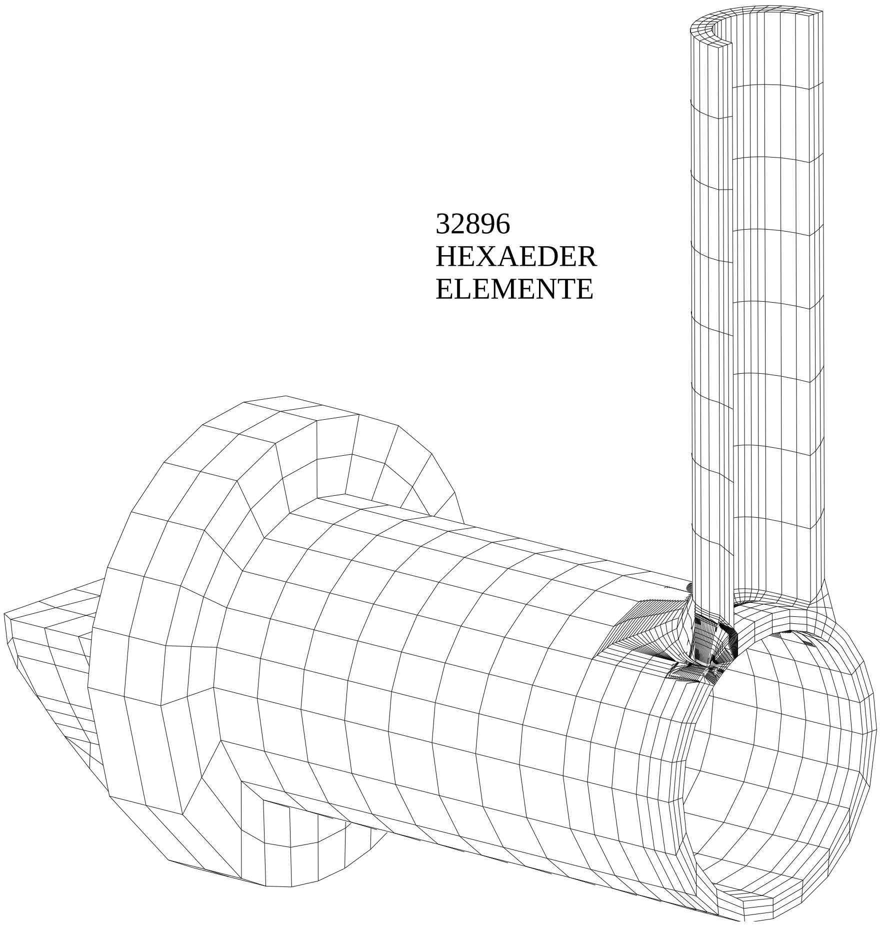
2D FE mesh of a biaxially loaded cruciform specimen with two-fold symmetry



Detail of the FE mesh of the cruciform specimen at the crack tip with a regular arrangement of elements



3D FE mesh of a tubular joint under 3-point bending, having one symmetry plane



Detail of the FE mesh of the semi-elliptical surface flaw in the weldment of the tubular joint