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Final Draft
of the original manuscript:

Mosler, J.; Bruhns, O.T.:

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In: Computer Methods in Applied Mechanics and Engineering (2009) Elsevier

DOI: 10.1016/j.cma.2009.07.006

On the implementation of rate-independent standard dissipative solids at finite strain – Variational constitutive updates

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Abstract

This paper is concerned with an efficient, variationally consistent, implementation for rate-independent dissipative solids at finite strain. More precisely, focus is on finite strain plasticity theory based on a multiplicative decomposition of the deformation gradient. Adopting the formalism of standard dissipative solids which allows to describe constitutive models by means of only two potentials being the Helmholtz energy and the yield function (or equivalently, a dissipation functional), finite strain plasticity is recast into an equivalent minimization problem. In contrast to previous models, the presented framework covers isotropic and kinematic hardening as well as isotropic and anisotropic elasticity and yield functions. Based on this approach a novel numerical implementation representing the main contribution of the paper is given. Analogously to the theoretical part, the algorithmic formulation is variationally consistent, i.e., all unknown variables follow naturally from minimizing the energy of the considered system. Extending previously published works on these methods, the advocated model does not rely on any material symmetry and therefore, it can be applied to a broad range of different plasticity theories. As an example, an anisotropic Hill-type model is developed. Numerical examples demonstrate the applicability and the performance of the proposed implementation.

Key words: Variational principles, Energy minimization, Variational constitutive updates, Standard dissipative solids
PACS:

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1 Introduction

Nowadays, computational plasticity represents an indispensable tool for the design of complex engineering structures. Considering a certain time interval $[t_n; t_{n+1}]$, the goal of computational plasticity is the calculation of all state and history variables $\boldsymbol{X}$ at time t_{n+1}, i.e., $\boldsymbol{X}_n \rightarrow \boldsymbol{X}_{n+1}$. Such methods are usually based on a time integration transforming the underlying differential equations such as the evolution laws into a set of non-linear equations. Subsequently, the resulting algebraic problem is solved iteratively. A typical example is given by the return-mapping algorithm, cf. [1,2]. This well-established first-order scheme consists of an (implicit) backward-Euler integration combined with a Newton-iteration. By now, the return-mapping algorithm can be considered as a state-of-the-art method even for the geometrically exact framework (large strains), cf. [3–5]. Most frequently, the underlying mechanical models are based on a multiplicative decomposition of the deformation gradient ($\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}$). Although many fundamental problems in finite strain elastoplasticity are still unanswered (see, e.g. [6,7]), the kinematical assumption $\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}$ is also made in the present work.

Mathematically, plasticity theory represents a non-smooth and highly non-linear problem in general. Clearly, the non-smoothness is a direct consequence of the elastic-plastic transition, while non-linearity results from the evolution equations, cf. [1,2]. Even worse, many plasticity models lead to non-unique solutions. For instance, crystal plasticity theory in the sense of Schmid shows this problem (see [8]). Due to these aforementioned issues, computational plasticity, although already established in the 60s (see [9]), is far from being completely solved and the development of an efficient and robust numerical implementation is not straightforward at all.

If the evolution equations and the flow rule obey the so-called normality rule (associative models), they can be elegantly derived from a variational principle – the postulate of maximum dissipation, cf. [10]. Comi and co-workers realized that by recourse to time discretization, a similar variational concept can be derived even for the discrete setting, see [11,12]. In those works, the authors, derived a Hu-Washizu functional whose minimum corresponds to the solution of the discretized algebraic differential equations defining the material model. In the respective numerical implementation, the constitutive model was enforced in a weak sense. That is, the resulting finite element formulation is different compared to the one usually applied in computational plasticity, cf. [2,1], i.e., a pointwise description (usually at the integration points).

Probably inspired by the works [11,12], Ortiz and co-workers advocated a constitutive update based on a minimization principle as well, cf. [13–15]. Nevertheless, in contrast to the previous works, the proposed algorithmic formulation coincides with the structure of standard finite element codes. More precisely, the update is performed pointwise at the integration points. Similar numerical procedure and further elaborations can be found, for instance, in [16–19]. For models based on linearized kinematics, the reader is referred to [20].

The advantages resulting from such a variational constitutive update are manifold. On the one hand, the existence of solutions can be analyzed by using the same tools originally designed for hyperelastic material models, cf. [21,15,16]. On the other hand, a minimum principle can be taken as a canonical basis for error estimation, cf. [14,22–24]. In addition, from an implementational point of view, a minimization principle opens up the possibility to apply state of the art optimization algorithms. Particularly for multisurface plasticity models such as single-crystal plasticity this represents an interesting feature.

In the present paper, an enhanced constitutive update for so-called standard dissipative solids obeying the postulate of maximum dissipation, in line with [13,14,16,20,17–19], is elaborated. In contrast to the algorithms discussed in the cited works, the advocated novel scheme does not rely on any material symmetry and therefore, it can be applied to a broad range of different plasticity theories. More precisely, arbitrary material symmetries concerning the elastic response and the yield function can be taken into account. Furthermore, kinematic and coupled isotropic/kinematic hardening are covered by the novel algorithmic formulation. As an example, a fully anisotropic Hill-type model (orthotropic elastic response and orthotropic yield function) is implemented.

The paper is organized as follows: Section 2 is concerned with a concise review of finite strain plasticity theory based on a multiplicative decomposition of the deformation gradient ($\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}$). While Subsection 2.1 covers the fundamentals, the variational structure of plasticity associated with so-called standard dissipative solids is discussed in Subsection 2.2. Section 2 is completed by a relatively complex example being a fully anisotropic Hill-type model (Subsection 2.3). The main contribution of the present paper dealing with a novel numerical implementation suitable for standard dissipative solids at finite strain is addressed in Section 3. The underlying key idea is to conveniently parameterize the restrictions imposed by the flow and the hardening rules. Finally, the performance and the robustness of the resulting algorithmic formulation are demonstrated by means of selected numerical examples (Section 4).

2 Finite strain plasticity theory

The fundamentals of a variationally consistent finite strain plasticity theory based on a multiplicative decomposition of the deformation gradient in the sense of [25] are briefly discussed in this section. For the sake of simplicity, isothermal static conditions are assumed. For modeling a dissipative material response, a description with internal state variables is used, cf. [26]. While Subsection 2.1 is concerned with a concise review of conventional plasticity theory at finite strains, standard dissipative solids are addressed in Subsection 2.2. These models allow to describe plasticity theories by means of only two independent functionals being the Helmholtz energy and the yield function (or equivalently, a dissipation functional). The present section is completed by a variationally consistent Hill-type plasticity model which includes an anisotropic elastic energy and an anisotropic yield function, together with isotropic and kinematic hardening.

2.1 Fundamentals

Without going too much into detail and following Lee [25], a multiplicative decomposition of the deformation gradient $\boldsymbol{F} := \mathrm{GRAD}\boldsymbol{\varphi}$ into an elastic part $\boldsymbol{F}^{\mathrm{e}}$ and a plastic part $\boldsymbol{F}^{\mathrm{p}}$ of the type

$$\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \cdot \boldsymbol{F}^{\mathrm{p}}, \qquad \text{with} \qquad \det \boldsymbol{F}^{\mathrm{e}} > 0,\ \det \boldsymbol{F}^{\mathrm{p}} > 0 \tag{1}$$

is adopted. For a comprehensive overview and critical comments on different plasticity formulations at finite strains, refer to [27,28,7]. Based on the split (1), the Helmholtz energy of the considered solid can be written as

$$\Psi = \Psi(\boldsymbol{F}^{\mathrm{e}}, \boldsymbol{F}^{\mathrm{p}}, \boldsymbol{\alpha}) \tag{2}$$

see [29,30,1,2]. Here, $\boldsymbol{\alpha} \in \mathbb{R}^n$ is a collection of strain-like internal variables associated with hardening or softening. Assuming that the elastic response modeled by $\bar{\Psi}^{\mathrm{e}}$ is completely independent of the internal processes reproduced by $\boldsymbol{\alpha}$, an energy functional Ψ of the type

$$\Psi = \bar{\Psi}^{\mathrm{e}}(\boldsymbol{F}^{\mathrm{e}}) + \Psi^{\mathrm{p}}(\boldsymbol{\alpha}) \tag{3}$$

is adopted. Clearly, by the principle of material frame indifference, $\bar{\Psi}^{\mathrm{e}}(\boldsymbol{F}^{\mathrm{e}}) = \Psi^{\mathrm{e}}(\boldsymbol{C}^{\mathrm{e}})$ where $\boldsymbol{C}^{\mathrm{e}} := \boldsymbol{F}^{\mathrm{e}^T} \cdot \boldsymbol{F}^{\mathrm{e}}$ is the elastic right Cauchy-Green tensor. The second term in Eq. (3), denoted as Ψ^{p}, represents the stored energy due to plastic work. It is associated with isotropic/kinematic hardening/softening. For more details about energy functionals of the type (3), refer to [31]. It should be noted that in most applications, a functional of the type (3) is chosen.

Considering an isothermal process, the second law of thermodynamics yields

$$\mathcal{D} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\Psi} = \boldsymbol{S} : \frac{1}{2}\,\dot{\boldsymbol{C}} - \dot{\Psi} \geq 0 \tag{4}$$

and finally, by using Eq. (1) and (3) the dissipation $\mathcal{D}$ reads

$$\mathcal{D} = \left(\boldsymbol{F}^{\mathrm{p}} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{p^T} - 2\, \frac{\partial \Psi}{\partial \boldsymbol{C}^{\mathrm{e}}} \right) : \frac{1}{2}\, \dot{\boldsymbol{C}}^{\mathrm{e}} + \boldsymbol{S} : \left(\boldsymbol{F}^{\mathrm{p}^T} \cdot \boldsymbol{C}^{\mathrm{e}} \cdot \dot{\boldsymbol{F}}^{\mathrm{p}} \right) + \boldsymbol{Q} \cdot \dot{\boldsymbol{\alpha}} \geq 0. \quad (5)$$

In Eqs. (4) and (5), $\boldsymbol{P}$ and $\boldsymbol{S} := \boldsymbol{F}^{-1} \cdot \boldsymbol{P}$ denote the first and the second Piola-Kirchhoff stress tensor and $\boldsymbol{Q} := -\partial_{\boldsymbol{\alpha}} \Psi$ is the stress-like internal variable work conjugate to $\boldsymbol{\alpha}$. According to Ineq. (5), the dissipation is decomposed additively into one part associated with the elastic strain rate and a second part corresponding to plastic deformation. Since both parts are independent of one another, Ineq. (5) gives rise to

$$\boldsymbol{S} = 2\, \frac{\partial \Psi}{\partial \boldsymbol{C}} = 2\, \boldsymbol{F}^{\mathrm{p}^{-1}} \cdot \frac{\partial \Psi}{\partial \boldsymbol{C}^{\mathrm{e}}} \cdot \boldsymbol{F}^{\mathrm{p}^{-T}} \quad (6)$$

and the reduced dissipation inequality

$$\mathcal{D} = \boldsymbol{\Sigma} : \boldsymbol{L}^{\mathrm{p}} + \boldsymbol{Q} \cdot \dot{\boldsymbol{\alpha}} \geq 0. \quad (7)$$

Here and henceforth, $\boldsymbol{\Sigma} = 2\ \boldsymbol{C}^{\mathrm{e}} \cdot \partial_{\boldsymbol{C}^{\mathrm{e}}} \Psi$ are the Mandel stresses (cf. [32]) and $\boldsymbol{L}^{\mathrm{p}} = \dot{\boldsymbol{F}}^{\mathrm{p}} \cdot \boldsymbol{F}^{\mathrm{p}^{-1}}$ denotes the plastic velocity gradient. It bears emphasis that both objects belong to the intermediate configuration. This is sometimes highlighted by using overlined letters.

Next, the elastic domain has to be defined. For that purpose, the admissible stress space $\mathbb{E}_{\boldsymbol{\sigma}}$ is introduced, cf. [29]. Since according to Ineq. (7), the reduced dissipation inequality depends naturally on the Mandel stresses, $\mathbb{E}_{\boldsymbol{\sigma}}$ is formulated in terms of $\boldsymbol{\Sigma}$, i. e.,

$$\mathbb{E}_{\boldsymbol{\sigma}} = \left\{ (\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathbb{R}^{9+n} \;\middle|\; \phi(\boldsymbol{\Sigma}, \boldsymbol{Q}) \leq 0 \right\}. \quad (8)$$

The boundary $\partial \mathbb{E}_{\boldsymbol{\sigma}}$ represents a level set function measuring the elastic limit of the considered material. That is, if $(\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathrm{int}\mathbb{E}_{\boldsymbol{\sigma}}$, the solid deforms purely elastically. Only if $(\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \partial \mathbb{E}_{\boldsymbol{\sigma}}$, a plastic response is possible. Clearly, the *yield function* ϕ has to be derived from experimental observation. Additionally, ϕ must be convex and sufficiently smooth, cf. [33]. The constitutive model is completed by evolution equations for $\boldsymbol{L}^{\mathrm{p}}$ and $\boldsymbol{\alpha}$ and by loading/unloading conditions. They can be naturally derived from the postulate of maximum dissipation. More precisely,

$$\max_{(\tilde{\boldsymbol{\Sigma}}, \tilde{\boldsymbol{Q}}) \in \mathbb{E}_{\boldsymbol{\sigma}}} \left[\tilde{\boldsymbol{\Sigma}} : \boldsymbol{L}^{\mathrm{p}} + \tilde{\boldsymbol{Q}} \cdot \dot{\boldsymbol{\alpha}}] \right]. \quad (9)$$

This postulate leads to the evolution equations

$$\boldsymbol{L}^{\mathrm{p}} = \lambda\, \partial_{\boldsymbol{\Sigma}} \phi \qquad \dot{\boldsymbol{\alpha}} = \lambda\, \partial_{\boldsymbol{Q}} \phi, \quad (10)$$

together with the Karush-Kuhn-Tucker conditions

$$\lambda \geq 0 \qquad \phi\,\lambda \geq 0. \tag{11}$$

As a result, plastic deformations require $(\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \partial\mathbb{E}_{\boldsymbol{\sigma}}$. The plastic multiplier λ is obtained from the consistency condition

$$\dot{\phi} = 0. \tag{12}$$

Evolution laws of the type (10) are characterized by the property that the rates of the internal variables (together with $\boldsymbol{L}^{\mathrm{p}}$) are proportional to the gradient of the yield function. Clearly, such laws are referred to as *associated flow rules* or *normality rules*.

2.2 Standard dissipative solids

In this section, the plasticity framework discussed before is recast into an equivalent minimization problem. More precisely, the goal of this section is the derivation of a potential, from which the unknown deformation mapping can be computed by minimization. Evidently, for path-dependent problems such as plasticity theory, this potential is defined pointwise (with respect to the (pseudo) time). As in the previous sections, isothermal conditions are assumed and dynamical effects are neglected. This section follows to a large extent [13,16].

In line with [13,16], the functional

$$\tilde{\mathcal{E}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}, \boldsymbol{Q}) = \dot{\Psi}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) + \mathcal{D}(\dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}, \boldsymbol{Q}) + J(\boldsymbol{\Sigma}, \boldsymbol{Q}) \tag{13}$$

is introduced. Here, J is the characteristic function of $\mathbb{E}_{\boldsymbol{\sigma}}$, i.e.,

$$J(\boldsymbol{\Sigma}, \boldsymbol{Q}) := \begin{cases} 0 & \forall (\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathbb{E}_{\boldsymbol{\sigma}} \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

As a result, for admissible stress states, i. e., $(\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathbb{E}_{\boldsymbol{\sigma}}$, $\tilde{\mathcal{E}}$ represents the sum of the rate of the stored energy and the dissipation. More precisely, if $(\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathbb{E}_{\boldsymbol{\sigma}}$,

$$\tilde{\mathcal{E}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}, \boldsymbol{Q}) = \boldsymbol{P} : \dot{\boldsymbol{F}} =: \mathcal{P}. \tag{15}$$

That is, $\tilde{\mathcal{E}}$ equals the stress power denoted as $\mathcal{P}$. Inadmissible stress states are penalized by $J = \infty$. The interesting properties of the functional (13) become apparent, if the stationarity conditions are computed. A variation of $\tilde{\mathcal{E}}$ with respect to $(\boldsymbol{\Sigma}, \boldsymbol{Q})$ leads to

$$(\boldsymbol{L}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) \in \partial J. \tag{16}$$

where ∂J is the sub-differential of J. The respective equation associated with $\dot{\boldsymbol{\alpha}}$ reads

$$\boldsymbol{Q} = -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}}. \tag{17}$$

Finally, a variation with respect to $\dot{\boldsymbol{F}}^{\mathrm{p}}$ yields

$$\boldsymbol{\Sigma} = \boldsymbol{F}^{\mathrm{e}T} \cdot \frac{\partial \Psi}{\partial \boldsymbol{F}^{\mathrm{e}}} = 2\,\boldsymbol{C}^{\mathrm{e}} \cdot \frac{\partial \Psi}{\partial \boldsymbol{C}^{\mathrm{e}}}. \tag{18}$$

As a consequence, the stationarity condition of $\tilde{\mathcal{E}}$ results in the flow rule (16), the constitutive relation for the internal stress-like variables (17) and the constitutive relation for the Mandel stresses $\boldsymbol{\Sigma}$. The remaining variation of $\tilde{\mathcal{E}}$ with respect to $\dot{\boldsymbol{\varphi}}$ will be discussed later.

According to [13,16], it is possible to derive a reduced functional, denoted as $\mathcal{E}$, which only depends on the rate of the deformation and the strain-like internal variables $\boldsymbol{\alpha}$ and $\boldsymbol{F}^{\mathrm{p}}$. For that purpose, $\mathcal{E}$ is re-written by applying the Legendre transformation

$$J^*(\bar{\boldsymbol{L}}^{\mathrm{p}}, \dot{\bar{\boldsymbol{\alpha}}}) = \sup \left\{ \boldsymbol{\Sigma} : \bar{\boldsymbol{L}}^{\mathrm{p}} + \boldsymbol{Q} \cdot \dot{\bar{\boldsymbol{\alpha}}} \;\middle|\; (\boldsymbol{\Sigma}, \boldsymbol{Q}) \in \mathbb{E}_{\boldsymbol{\sigma}} \right\} \tag{19}$$

of J. Since J^* is positively homogeneous of degree one, a maximization of $\tilde{\mathcal{E}}$ with respect to $(\boldsymbol{\Sigma}, \boldsymbol{Q})$, results in

$$\mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) = \dot{\Psi}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) + J^*(\dot{\boldsymbol{L}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}). \tag{20}$$

Hence, the only remaining variables are $\dot{\boldsymbol{\varphi}}$, $\dot{\boldsymbol{F}}^{\mathrm{p}}$ and $\dot{\boldsymbol{\alpha}}$. Even more importantly, the strain-like internal variables $\boldsymbol{F}^{\mathrm{p}}$ and $\boldsymbol{\alpha}$ follow jointly from the minimization principle

$$\overset{\circ}{\Psi}_{\mathrm{red}}\,(\dot{\boldsymbol{\varphi}}) := \inf_{\dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}} \mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) \tag{21}$$

which, itself, gives rise to the introduction of the reduced functional $\overset{\circ}{\Psi}_{\mathrm{red}}$ depending only on the deformation mapping. It is interesting to note that for hyperelastic continua, $\overset{\circ}{\Psi}_{\text{red}}$ equals the rate of the strain-energy density, i. e.,

$$\overset{\circ}{\Psi}_{\text{red}}\ (\dot{\boldsymbol{\varphi}}) = \dot{\Psi}(\boldsymbol{\varphi}). \tag{22}$$

As a result, in this case, $\overset{\circ}{\Psi}_{\text{red}}$ represents the time derivative of a potential. Further elaborating this analogy, it can be shown that the time integration of Eq. (21) or Eq. (22) over the interval $[t_n, t_{n+1}]$ defines an incremental potential which acts like a standard hyperelastic energy functional for the stresses. Hence, the first Piola-Kirchhoff type stresses are obtained from

$$\boldsymbol{P} = \frac{\partial \left(\int\limits_{t_n}^{t_{n+1}} \overset{\circ}{\Psi}_{\text{red}}\ (\dot{\boldsymbol{\varphi}})\ \mathrm{d}t \right)}{\partial \boldsymbol{F}}. \tag{23}$$

It bears emphasis that if $\overset{\circ}{\Psi}_{\text{red}}$ represents the time derivative of a potential, the standard hyperelastic relation is recovered, i.e., $\boldsymbol{P} = \partial_{\boldsymbol{F}}\Psi$. Applying Eq. (23) it is relatively straightforward to extend the principle of minimum potential energy to standard dissipative solids. More precisely, the unknown deformation mapping $\boldsymbol{\varphi}$ follows from the minimization principle

$$\boldsymbol{\varphi} = \arg \inf_{\boldsymbol{\varphi}} I_{\text{inc}}(\boldsymbol{\varphi}). \tag{24}$$

with the incremental potential $I_{\text{inc}}(\boldsymbol{\varphi})$ being defined as

$$I_{\text{inc}}(\boldsymbol{\varphi}) = \inf_{\boldsymbol{F}^{\text{p}}, \boldsymbol{\alpha}} \left[\int\limits_{\Omega} \int\limits_{t_n}^{t_{n+1}} \mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\text{p}}, \dot{\boldsymbol{\alpha}})\ \mathrm{d}t\ \mathrm{d}V - \int\limits_{\Omega} \rho_0\ \boldsymbol{B} \cdot \boldsymbol{\varphi}\ \mathrm{d}V - \int\limits_{\partial_2\Omega} \bar{\boldsymbol{T}} \cdot \boldsymbol{\varphi}\ \mathrm{d}A \right]. \tag{25}$$

Here, ρ_0, $\boldsymbol{B}$ and $\bar{\boldsymbol{T}}$ denote the referential density, prescribed body forces and prescribed surface forces acting at $\partial_2\Omega$.

For a more detailed derivation, the interested reader is referred to [13,16]. It should be pointed out that the minimization path of the internal variables according to Problem (25) can only be computed analytically for selected, relatively simple examples, cf. [16]. Hence, an efficient numerical implementation as proposed in the following section is required in general.

2.3 Prototype model

For the sake of concreteness, a prototype model falling into the range of the aforementioned standard dissipative solids is given. Since the main contribution of the present paper is the derivation of an efficient implementation which holds for a broad range of constitutive models including elastic and plastic anisotropy and kinematic hardening, a model combining all these physical phenomena is considered. In the case of a fully isotropic von Mises plasticity formulation or single slip systems, refer to [13,16].

The first component of the model is the stored energy functional. In line with Eq. (3), the Helmholtz energy

$$\Psi(\boldsymbol{F}^{\mathrm{e}}, \alpha_{\mathrm{i}}, \boldsymbol{\alpha}_{\mathrm{k}}) := \Psi^{\mathrm{e}}(\boldsymbol{F}^{\mathrm{e}}) + \Psi^{\mathrm{p}}(\alpha_{\mathrm{i}}, \boldsymbol{\alpha}_{\mathrm{k}}) \tag{26}$$

is further decomposed into the part $\Psi^{\mathrm{p}}_{\mathrm{i}}$ associated with isotropic hardening, while $\Psi^{\mathrm{p}}_{\mathrm{k}}$ corresponds to kinematic hardening, i.e.,

$$\Psi^{\mathrm{p}}(\boldsymbol{\alpha}_{\mathrm{k}}) = \Psi^{\mathrm{p}}_{\mathrm{i}}(\alpha_{\mathrm{i}}) + \Psi^{\mathrm{p}}_{\mathrm{k}}(\boldsymbol{\alpha}_{\mathrm{k}}). \tag{27}$$

Here, the internal strain-like variables α_{i} and $\boldsymbol{\alpha}_{\mathrm{k}}$ are scalar-valued and second-order tensors, respectively. It bears emphasis that no assumption concerning the elastic or the plasticity-induced isotropy has been made yet. Hence, for instance, by introducing structural tensors into $\Psi^{\mathrm{e}}(\boldsymbol{F}^{\mathrm{e}})$, elastic anisotropy can be taken into account. Furthermore, it is noteworthy that if $\Psi^{\mathrm{p}}_{\mathrm{k}}(\boldsymbol{\alpha}_{\mathrm{k}})$ is isotropic, its material time derivative reads

$$\dot{\Psi}^{\mathrm{p}}_{\mathrm{k}} = \partial_{\boldsymbol{\alpha}_{\mathrm{k}}} \Psi^{\mathrm{p}}_{\mathrm{k}} \cdot \dot{\boldsymbol{\alpha}}_{\mathrm{k}} = \partial_{\boldsymbol{\alpha}_{\mathrm{k}}} \Psi^{\mathrm{p}}_{\mathrm{k}} \cdot \overset{\circ}{\boldsymbol{\alpha}}_{\mathrm{k}} \tag{28}$$

with $\overset{\circ}{\boldsymbol{\alpha}}_{\mathrm{k}}$ being an arbitrary corotational (objective) time derivative, cf. [34]. This property can be useful for deriving objective evolution equations. While the choice of the isotropic response ($\Psi^{\mathrm{p}}_{\mathrm{i}}$) depending on experiments is uncomplicated, the part describing kinematic hardening is far from being trivial. For a detailed discussion about this issue in the context of a geometrically linearized theory, refer to [35]. For the sake of simplicity and even more importantly, for the sake of interpretableness of the numerical results reported in Section 4, linear hardening is considered, i.e.,

$$\Psi^{\mathrm{p}}_{\mathrm{i}}(\alpha_{\mathrm{i}}) = \frac{1}{2}\, H_{\mathrm{i}}\, \alpha_{\mathrm{i}}^2, \qquad \Psi^{\mathrm{p}}_{\mathrm{k}}(\boldsymbol{\alpha}_{\mathrm{k}}) = \frac{1}{2}\, H_{\mathrm{k}}\, ||\boldsymbol{\alpha}_{\mathrm{k}}||^2 \tag{29}$$

However, this assumption is not crucial for the numerical implementation presented in the following section. Thus, the algorithmic formulation covers more complicated hardening models as well.

The next component of the prototype model is the yield function spanning the admissible elastic domain. Here, a Hill-type model defined by

$$\phi(\boldsymbol{\Sigma}, \boldsymbol{Q}_{\mathrm{k}}, Q_{\mathrm{i}}) := \Sigma^{\mathrm{eq}}(\boldsymbol{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) - Q_{\mathrm{i}} - Q^{\mathrm{eq}}_{o} \tag{30}$$

together with the equivalent stress

$$\Sigma^{\mathrm{eq}}(\boldsymbol{A}) := \sqrt{\boldsymbol{A} : \mathbb{H} : \boldsymbol{A}} \tag{31}$$

is chosen. With

$$\mathbb{P}_{\mathrm{Dev}} := \mathbb{I} - \frac{1}{3}\, \mathbf{1} \otimes \mathbf{1} \tag{32}$$

denoting the 4th-order tensor mapping an arbitrary second-order tensor $\boldsymbol{A}$ onto its deviatoric counterpart $\mathrm{Dev}[\boldsymbol{A}] = \mathbb{P}_{\mathrm{Dev}} : \boldsymbol{A} = \boldsymbol{A} - 1/3\ \mathrm{tr}(\boldsymbol{A})\ \mathbf{1}$, the 4th-order tensor $\mathbb{H}$ is defined by

$$\mathbb{H} = \mathbb{P}_{\mathrm{Dev}} : \mathbb{D} : \mathbb{P}_{\mathrm{Dev}}. \tag{33}$$

Clearly, by setting $\mathbb{D} = \mathbb{I}$, the identity $\mathbb{H} = \mathbb{P}_{\mathrm{Dev}}$ is obtained yielding $\Sigma^{\mathrm{eq}}(\boldsymbol{A}) = \sqrt{\mathrm{Dev}[\boldsymbol{A}] : \mathrm{Dev}[\boldsymbol{A}]}$. Hence, standard von Mises plasticity theory is included within the prototype model. For anisotropic yield functions, $\mathbb{D}$ does not equal the identity anymore, but its components have to be related to the yield stresses in different directions. However, it bears emphasis that $\mathbb{D}$ cannot be chosen arbitrarily. For instance, it has to be guaranteed that the yield function is convex implying that $\mathbb{H}$ is (semi-) positive definite.

Based on the yield function (30), the evolution equations defining the standard dissipative solid are given by

$$\begin{aligned} \boldsymbol{L}^{\mathrm{p}} := \dot{\boldsymbol{F}}^{\mathrm{p}} \cdot \boldsymbol{F}^{\mathrm{p}-1} &= \lambda\, \frac{\mathbb{H} : \boldsymbol{\Sigma}}{\Sigma^{\mathrm{eq}}} \\ \dot{\alpha}_{\mathrm{i}} &= \lambda\, \partial_{Q_{\mathrm{i}}} \phi \\ \dot{\boldsymbol{\alpha}}_{\mathrm{k}} &= \lambda\, \partial_{\boldsymbol{Q}_{\mathrm{k}}} \phi = -\boldsymbol{L}^{\mathrm{p}} \end{aligned} \tag{34}$$

and the Helmholtz energy (26) yields the thermodynamic forces

$$\begin{aligned} Q_{\mathrm{i}} &:= -\frac{\partial \Psi}{\partial \alpha_{\mathrm{i}}} = -H_{\mathrm{i}}\ \alpha_{\mathrm{i}} \\ \boldsymbol{Q}_{\mathrm{k}} &:= -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}_{\mathrm{k}}} = -H_{\mathrm{k}}\ \boldsymbol{\alpha}_{\mathrm{k}}. \end{aligned} \tag{35}$$

where Q_{i} and $\boldsymbol{Q}_{\mathrm{k}}$ are stress-like internal variables work conjugate to α_{i} and $\boldsymbol{\alpha}_{\mathrm{k}}$, respectively. Furthermore, since Σ^{eq} is a positively homogeneous function of degree one, the dissipation simplifies to

$$
\begin{aligned}
\mathcal{D} = \mathbf{\Sigma} : \boldsymbol{L} - \dot{\Psi} &= \mathbf{\Sigma} : \boldsymbol{L}^{\mathrm{p}} - \dot{\Psi}^{\mathrm{p}} \\
&= \lambda \, \left[\partial_{\mathbf{\Sigma}}\phi : (\mathbf{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) - Q_{\mathrm{i}}\right] \\
&= \lambda \, \left[\Sigma^{\mathrm{eq}}(\mathbf{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) - Q_{\mathrm{i}}\right] \\
&= \lambda \, Q_{o}^{\mathrm{eq}} \geq 0.
\end{aligned}
\tag{36}
$$

Thus, the second law of thermodynamics is indeed fulfilled and even more importantly, the dissipation can be computed explicitly.

3 Numerical implementation

This section representing the main contribution of the present paper is concerned with a novel numerical implementation suitable for standard dissipative solids at finite strains. Analogously to the previous section, the algorithmic formulation is variationally consistent, i.e., all unknown variables follow naturally from minimizing the energy of the considered system. In contrast to previous works on such methods, the advocated model does not rely on any material symmetry and therefore, it can be applied to a broad range of different plasticity theories. This section is organized as follows: In Subsection 3.1 a time discretization transforming the continuous minimization problem (21) into its discrete counterpart is briefly presented. Based on this approximation and as a motivation, the numerical implementation for relatively simple prototype models such as von Mises plasticity theory is carefully analyzed first, see Subsection 3.2. The underlying key idea is to conveniently parameterize the restrictions imposed by the flow rule. Finally, these prototypes are generalized for more complex, possibly anisotropic, plasticity models in Subsection 3.3. An adapted implementation for fully isotropic model is discussed as well.

3.1 Time integration

One of the key ideas of the variationally consistent implementation of standard dissipative solids leading to so-called *variational constitutive updates* is the transformation of the continuous optimization problem (21) into a discrete counterpart. Conceptually, if a consistent time integration is applied, the integrated evolution equations are obtained from the minimization problem

$$
(\boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{\mathrm{k}}|_{n+1}, \alpha_{\mathrm{i}}|_{n+1}) = \arg \inf_{\boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{\mathrm{k}}|_{n+1}, \alpha_{\mathrm{i}}|_{n+1}} I_{\mathrm{inc}} \tag{37}
$$

with

$$I_{\text{inc}}(\boldsymbol{F}^{\text{p}}_{n+1}, \boldsymbol{\alpha}_{\text{k}}|_{n+1}, \alpha_{\text{i}}|_{n+1}) := \int\limits_{t_n}^{t_{n+1}} \mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\text{p}}, \dot{\boldsymbol{\alpha}}) \, \text{d}t = \Psi_{n+1} - \Psi_n + \int\limits_{t_n}^{t_{n+1}} J^* \, \text{d}t \tag{38}$$

cf. Eq. (21). Clearly, I_{inc} depends additionally on the (known) previous time step $(\boldsymbol{F}^{\text{p}}_n, \boldsymbol{\alpha}_{\text{k}}|_n, \alpha_{\text{i}}|_n)$ as well as on the (given) deformation gradient $\boldsymbol{F}_{n+1}$. However, this is not highlighted explicitly.

In line with numerical implementations for standard (not variationally consistent) finite strain plasticity models such as [2,1], Eq. (37) is approximated by applying a consistent time discretization to the evolution equations. For the sake of concreteness, a first-order fully implicit scheme is adopted. More precisely, with the notation

$$\Delta\lambda := \int\limits_{t_n}^{t_{n+1}} \lambda \, \text{d}t \geq 0 \tag{39}$$

the following approximations are used:

$$\begin{aligned} \boldsymbol{F}^{\text{p}}_{n+1} &= \exp\left[\Delta\lambda \;\; \partial_{\boldsymbol{\Sigma}}\phi|_{n+1}\right] \cdot \boldsymbol{F}^{\text{p}}_n \\ \alpha_{\text{i}}\big|_{n+1} &= \alpha_{\text{i}}\big|_n + \Delta\lambda \, \partial_{Q_{\text{i}}}\phi \\ \boldsymbol{\alpha}_{\text{k}}\big|_{n+1} &= \boldsymbol{\alpha}_{\text{k}}\big|_n + \Delta\lambda \;\; \partial_{\boldsymbol{Q}_{\text{k}}}\phi\Big|_{n+1} . \end{aligned} \tag{40}$$

As an alternative to Eq $(40)_c$, the tensor-valued internal variables $\boldsymbol{\alpha}_{\text{k}}$ could be integrated by applying the exponential map, cf. [36], leading to

$$\boldsymbol{\alpha}_{\text{k}}\big|_{n+1} = \boldsymbol{\alpha}_{\text{k}}\big|_n \cdot \exp\left[\Delta\lambda \;\; \partial_{\boldsymbol{Q}_{\text{k}}}\phi\Big|_{n+1} \cdot \boldsymbol{\alpha}_{\text{k}}^{-1}\Big|_{n+1}\right] . \tag{41}$$

Clearly, with Eq. $(40)_a$, the elastic part of the deformation gradient reads

$$\boldsymbol{F}^{\text{e}}_{n+1} = \boldsymbol{F}^{\text{e}}_{\text{trial}} \cdot \exp\left[-\Delta\lambda \, \partial_{\boldsymbol{\Sigma}}\phi\right] \qquad \boldsymbol{F}^{\text{e}}_{\text{trial}} := \boldsymbol{F}_{n+1} \cdot (\boldsymbol{F}^{\text{p}}_n)^{-1}. \tag{42}$$

It bears emphasis that the time integrations (39)–(41) are indeed consistent and hence, convergence to the analytical solution is guaranteed.

Inserting Eqs. (39), (40) and (42) into Eq. (38), the time integration of I_{inc} yields

$$\begin{aligned} I_{\text{inc}}(\boldsymbol{F}^{\text{p}}_{n+1}, \boldsymbol{\alpha}_{\text{k}}|_{n+1}, \alpha_{\text{i}}|_{n+1}, \Delta\lambda) \approx \Psi_{n+1} - \Psi_n \; + \; \boldsymbol{\Sigma}_{n+1} : \log[\boldsymbol{F}^{\text{p}}_{n+1} \cdot (\boldsymbol{F}^{\text{p}}_n)^{-1}] \\ + \; [\boldsymbol{Q}_{\text{k}} : \partial_{\boldsymbol{Q}_{\text{k}}}\phi]|_{n+1} \, \Delta\lambda + [Q_{\text{i}} \; \partial_{Q_{\text{i}}}\phi]|_{n+1} \, \Delta\lambda \end{aligned} \tag{43}$$

Since the term Ψ_n shifting the energy depends only on the previous time step, it does not affect the optimization problem (37) and hence, it can be neglected. According to the derivation, the potential (43) depends on the considered time integration and hence, uniqueness is only obtained in the limiting case $\Delta t \to 0$.

So far, variational constitutive updates are relatively simple and hence, the respective implementation seems to be straightforward. Unfortunately, this is not the case. The reasons for that are manifold. For instance, a direct minimization of Ψ_{inc} with respect to $\boldsymbol{F}^{\mathrm{p}}_{n+1}$ is not admissible, since $\boldsymbol{F}^{\mathrm{p}}$ has to comply with physical constraints resulting from the flow rule (and of course, $\det \boldsymbol{F}^{\mathrm{p}} > 0$). Furthermore, if plastic loading is considered, the additional restriction $\phi = 0$ has to be enforced relating the stresses (and thus the strains) to the internal variables $\boldsymbol{\alpha}_{\mathrm{k}}$ and α_{i}. Fortunately, all these problems can be solved efficiently by elaborating a suitable parameterization of the unknown variables. This will be shown in the next subsection.

Remark 1 *No physical assumption regarding the yield function ϕ has been made yet. In most applications ϕ is chosen to be of the type*

$$\phi(\boldsymbol{\Sigma}, \boldsymbol{Q}_{\mathrm{k}}, Q_{\mathrm{i}}) = \Sigma_{\mathrm{eq}}(\boldsymbol{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) - Q_{\mathrm{i}} - Q_0^{\mathrm{eq}}. \tag{44}$$

As a result, $\boldsymbol{Q}_{\mathrm{k}}$ represents a back-stress and furthermore, the identity

$$\dot{\boldsymbol{\alpha}}_{\mathrm{k}} = -\boldsymbol{L}^{\mathrm{p}}. \tag{45}$$

is fulfilled. Additionally, Σ^{eq} is often represented by a positively homogeneous function of degree one. Combining these physically sound constraints, the dissipation simplifies significantly, i.e.,

$$\mathcal{D} = \lambda\, Q_0^{\mathrm{eq}} \geq 0 \tag{46}$$

cf. Eq. (36) and finally,

$$I_{\mathrm{inc}}(\boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{\mathrm{k}}|_{n+1}, \alpha_{\mathrm{i}}|_{n+1}) \approx \Psi_{n+1} - \Psi_n + Q_0^{\mathrm{eq}}\, \Delta\lambda. \tag{47}$$

For instance, the Hill-type model presented in Subsection 2.3 complies with the aforementioned assumption.

3.2 Motivation: Implementation of some prototype models

As mentioned before, the main issue associated with variational constitutive updates is the numerical implementation of the minimization problem (37). It depends crucially on a suitable parameterization of the evolution equations. For a better understanding, the algorithmic formulation is briefly presented for four different prototype models. Each of those fulfills the restrictions summarized in Remark 1. Hence, the functional to be minimized is given by Eq. (47) and the constraint $\phi = 0$ is already included within the optimization, cf. Eq. (36).

3.2.1 Example: Single crystal plasticity

Since single-crystal plasticity (in the sense of Schmid's law) is based on associative evolution equations, the model is defined completely by the respective yield function ϕ. Introducing a slip plane by its corresponding normal vector $\bar{\boldsymbol{n}}$ and the slip direction $\bar{\boldsymbol{m}}$, ϕ is given by

$$\phi(\boldsymbol{\Sigma}, \alpha_{\mathrm{i}}) = |\boldsymbol{\Sigma} : (\bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}})\,| - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}}. \tag{48}$$

Evidently, the vectors $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ are objects that belong to the intermediate configuration. They are orthogonal to one another and time-independent, i. e.,

$$\bar{\boldsymbol{n}} \cdot \bar{\boldsymbol{m}} = 0 \qquad ||\bar{\boldsymbol{n}}||_2 = ||\bar{\boldsymbol{m}}||_2 = 1. \tag{49}$$

Based on Eq. (48), the evolution equations

$$\boldsymbol{L}^{\mathrm{p}} = \tilde{\lambda}\ (\bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}})\,, \qquad \text{with} \qquad \tilde{\lambda} = \lambda \operatorname{sign}\left[\boldsymbol{\Sigma} : (\bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}})\right], \qquad \dot{\alpha}_{\mathrm{i}} = -\lambda \tag{50}$$

are obtained. Hence, only one single variable being λ is unknown. Thus, a suitable parameterization of the time discretized evolution equations reads

$$\boldsymbol{F}^{\mathrm{p}}_{n+1} = (\boldsymbol{1} + \Delta\tilde{\lambda}\ \bar{\boldsymbol{m}} \otimes \bar{\boldsymbol{n}}) \cdot \boldsymbol{F}^{\mathrm{p}}_{n}, \qquad \alpha_{\mathrm{i}}|_{n+1} = \alpha_{\mathrm{i}}|_n - |\Delta\tilde{\lambda}| \tag{51}$$

and consequently, the minimization problem (36) depends only on the scalar-valued variable $\Delta\tilde{\lambda}$, i.e.,

$$\Delta\tilde{\lambda} := \arg\inf I_{\mathrm{inc}}(\Delta\tilde{\lambda}). \tag{52}$$

It bears emphasis that this property even holds for elastically anisotropic models.

3.2.2 Example: von Mises plasticity theory

For the next prototype model a fully isotropic elastic response is considered. Hence, the Mandel stresses are symmetric. The investigated von Mises yield function including isotropic hardening is given by

$$\phi(\boldsymbol{\Sigma}, \alpha_{\mathrm{i}}) = ||\mathrm{Dev}[\boldsymbol{\Sigma}]|| - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}} \tag{53}$$

where Dev[$\boldsymbol{\Sigma}$] is the deviator of $\boldsymbol{\Sigma}$ (compare to Subsection (2.3)). Consequently, the evolution equations are computed as

$$\boldsymbol{L}^{\mathrm{p}} = \lambda\, \boldsymbol{M}, \qquad \dot{\alpha}_{\mathrm{i}} = -\lambda, \quad \text{with} \quad \boldsymbol{M} := \frac{\mathrm{Dev}[\boldsymbol{\Sigma}]}{||\mathrm{Dev}[\boldsymbol{\Sigma}]||}. \tag{54}$$

Note that the tensor $\boldsymbol{M}$ shows the properties

$$||\boldsymbol{M}|| = 1, \qquad \mathrm{tr}[\boldsymbol{M}] = 0. \tag{55}$$

Furthermore, if the elastic model is isotropic, the elastic trial strains

$$\boldsymbol{C}^{\mathrm{e}}_{\mathrm{trial}} := (\boldsymbol{F}^{\mathrm{e}}{}_{\mathrm{trial}})^T \cdot \boldsymbol{F}^{\mathrm{e}}{}_{\mathrm{trial}} = \sum_{i=1}^{3} \left(\lambda_i^{\mathrm{etrial}}\right)^2\ \boldsymbol{N}_i \otimes \boldsymbol{N}_i \tag{56}$$

are coaxial to the (unknown) elastic strains $\boldsymbol{C}^{\mathrm{e}}$ and thus, to the Mandel stresses. As a result, since Eq. (53) is an isotropic tensor function in $\boldsymbol{\Sigma}$, the eigenvectors of the unknown tensor $\boldsymbol{L}^{\mathrm{p}}$ (or $\boldsymbol{M}$) are known in advance, i.e.,

$$\boldsymbol{M} = \sum_{i=1}^{3} \lambda_i^M\ \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \quad \text{with} \quad \sum_{i=1}^{3} \left(\lambda_i^M\right)^2 = 1, \quad \sum_{i=1}^{3} \lambda_i^M = 0, \tag{57}$$

see constraints (55). This, in turn, implies that only two parameters are unknown: the plastic multiplier and one additional parameter defining $\boldsymbol{M}$. Two convenient parameterizations of the restrictions imposed by the flow rule are given below

- Parameterization I depending on $\Delta\lambda_1^{\mathrm{p}}$, $\Delta\lambda_2^{\mathrm{p}}$

$$\begin{aligned} \Delta\lambda\, \partial_{\boldsymbol{\Sigma}}\phi &=: \sum_{i=1}^{3} \Delta\lambda_i^{\mathrm{p}}\ \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \quad \text{with} \quad \Delta\lambda_3^{\mathrm{p}} = -\Delta\lambda_1^{\mathrm{p}} - \Delta\lambda_2^{\mathrm{p}} \\ \Rightarrow \Delta\alpha_{\mathrm{i}} &= \alpha_{\mathrm{i}}|_{n+1} - \alpha_{\mathrm{i}}|_n = -\sqrt{\sum_{i=1}^{3} (\Delta\lambda_i^{\mathrm{p}})^2} \end{aligned} \tag{58}$$

- Parameterization II depending on θ and a; ([37])

$$\begin{aligned} \lambda_i^M &= \sqrt{\frac{2}{3}}\ \sin\left[\frac{2\ \alpha_i\ \pi}{3} - \theta\right], \quad \alpha_i = 1, 2, 3 \quad \text{cf. Eq. (57)} \\ \Delta\lambda &= a^2 \geq 0 \end{aligned} \tag{59}$$

3.2.3 Example: Associative Drucker-Prager plasticity model

This model is defined by the yield function

$$\phi(\boldsymbol{\Sigma}, \alpha_{\mathrm{i}}) = \kappa \, \mathrm{tr}[\boldsymbol{\Sigma}] + ||\mathrm{Dev}[\boldsymbol{\Sigma}]|| - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}}. \tag{60}$$

with κ being a material parameter. Eq. (60) yields the associative evolution equations

$$\boldsymbol{L}^{\mathrm{p}} = \lambda \, \left(\kappa \, \mathbf{1} + \boldsymbol{M}\right), \quad \dot{\alpha}_{\mathrm{i}} = -\lambda \tag{61}$$

where $\boldsymbol{M}$ is given by Eq. (57). Clearly, if the elastic response is fully isotropic, $\boldsymbol{M}$ and λ can again conveniently be parameterized by Eq. (59).

3.2.4 Example: Rankine plasticity model

The final example is associated with Rankine plasticity theory. The respective yield function is postulated to be

$$\phi(\boldsymbol{\Sigma}, \alpha_{\mathrm{i}}) = \Sigma_{\mathrm{max}}(\boldsymbol{\Sigma}) - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}}. \tag{62}$$

Here, $\Sigma_{\mathrm{max}}(\boldsymbol{\Sigma})$ is the maximum principal Mandel stress. Considering a fully isotropic elastic behavior, Eq. (62) results in the normality evolution equations

$$\boldsymbol{L}^{\mathrm{p}} = \lambda \, \boldsymbol{N}_{\mathrm{max}} \otimes \boldsymbol{N}_{\mathrm{max}}, \quad \dot{\alpha}_{\mathrm{i}} = -\lambda. \tag{63}$$

with $\boldsymbol{N}_{\mathrm{max}}$ denoting the eigenvector corresponding to Σ_{max}. Note that $\boldsymbol{N}_{\mathrm{max}}$ is known in advance, if a fully isotropic model is chosen (and the Baker-Ericksen inequalities hold). Hence, the only unknown parameter is λ which can conveniently be parameterized by $\Delta\lambda = a^2$.

3.3 An efficient variational constitutive update

As shown in the previous subsections, the numerical computation of the optimization problem (43) strongly depends on a convenient parameterization of the evolution equations and the flow rule. Such a parameterization which can be applied to a broad range of different constitutive models will be discussed in Subsection 3.3.1. Subsequently, the simplifications holding for fully isotropic models (elastic and plastic) are addressed in Subsection 3.3.2.

3.3.1 The general case

As evident from the prototype models, a convenient parameterization of the flow rule and the evolution equations depends strongly on the considered plasticity model, i.e., the yield function. For deriving a parameterization which holds for a broad range of different models, the (unknown) arguments $\boldsymbol{F}^{\mathrm{p}}_{n+1}$, $\boldsymbol{\alpha}_{\mathrm{k}}|_{n+1}$, $\alpha_{\mathrm{i}}|_{n+1}$ and $\Delta\lambda$ entering the incrementally defined potential (43) are replaced by a more suitable representation. More precisely,

$$I_{\mathrm{inc}} = I_{\mathrm{inc}}(\boldsymbol{M}, \boldsymbol{H}_{\mathrm{k}}, H_{\mathrm{i}}, \Delta\lambda) \tag{64}$$

with

$$\boldsymbol{M} := \partial_{\boldsymbol{\Sigma}}\phi, \quad \boldsymbol{H}_{\mathrm{k}} := \partial_{\boldsymbol{Q}_{\mathrm{k}}}\phi, \quad H_{\mathrm{i}} := \partial_{Q_{\mathrm{i}}}\phi. \tag{65}$$

Accordingly, $\boldsymbol{M}$, $\boldsymbol{H}_{\mathrm{k}}$ and H_{i} are the flow direction, the kinematic hardening direction and the isotropic hardening gradient, respectively. Clearly, $\boldsymbol{M}$, $\boldsymbol{H}_{\mathrm{k}}$ and H_{i} cannot be chosen arbitrarily, but have to comply with the restrictions imposed by the constitutive model. For this reason, the aforementioned directions are parameterized as follows:

$$\boldsymbol{M} = \boldsymbol{M}(\tilde{\boldsymbol{\Sigma}}) := \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}} \tag{66}$$

$$\boldsymbol{H}_{\mathrm{k}} = \boldsymbol{H}_{\mathrm{k}}(\tilde{\boldsymbol{Q}}_{\mathrm{k}}) := \partial_{\boldsymbol{Q}_{\mathrm{k}}}\phi\Big|_{\tilde{\boldsymbol{Q}}_{\mathrm{k}}} \tag{67}$$

$$H_{\mathrm{i}} = H_{\mathrm{i}}(\tilde{Q}_{\mathrm{i}}) := \partial_{Q_{\mathrm{i}}}\phi|_{\tilde{Q}_{\mathrm{i}}} \tag{68}$$

$$\Delta\lambda = \Delta\lambda(a) := a^2 \tag{69}$$

Here, the unknowns $\tilde{\boldsymbol{\Sigma}}$, $\tilde{\boldsymbol{Q}}_{\mathrm{k}}$ and $\tilde{Q}_{\mathrm{i}}$ denote pseudo stresses, a pseudo backstress and a pseudo stress-like hardening variable. It has to be emphasized that these pseudo variables are not identical to their physical counterparts in general, i.e.,

$$\tilde{\boldsymbol{\Sigma}} \neq \boldsymbol{\Sigma}, \quad \tilde{\boldsymbol{Q}}_{\mathrm{k}} \neq \boldsymbol{Q}_{\mathrm{k}}, \quad \tilde{Q}_{\mathrm{i}} \neq Q_{\mathrm{i}}. \tag{70}$$

More precisely, the variables $\tilde{\boldsymbol{\Sigma}}$, $\tilde{\boldsymbol{Q}}_{\mathrm{k}}$ and $\tilde{Q}_{\mathrm{i}}$ only define the flow and hardening directions. Thus, they are related with their physical counterparts by

$$\boldsymbol{M}(\tilde{\boldsymbol{\Sigma}}) = \boldsymbol{M}(\boldsymbol{\Sigma}), \quad \boldsymbol{H}_{\mathrm{k}}(\tilde{\boldsymbol{Q}}_{\mathrm{k}}) = \boldsymbol{H}_{\mathrm{k}}(\boldsymbol{Q}_{\mathrm{k}}), \quad H_{\mathrm{i}}(\tilde{Q}_{\mathrm{i}}) = H_{\mathrm{i}}(Q_{\mathrm{i}}). \tag{71}$$

As a result and in contrast to the original parameterization, Eqs. (66)–(69) automatically fulfill the restrictions associated with the considered constitutive model. For instance, in case of von Mises plasticity,

$$\phi = ||\mathrm{Dev}\boldsymbol{\Sigma}|| - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}} \quad \Rightarrow \quad \boldsymbol{M}(\tilde{\boldsymbol{\Sigma}}) = \frac{\mathrm{Dev}[\tilde{\boldsymbol{\Sigma}}]}{||\mathrm{Dev}[\tilde{\boldsymbol{\Sigma}}]||} \tag{72}$$

and thus the constraints,

$$\mathrm{tr}[\boldsymbol{M}] = 0, \qquad \boldsymbol{M} : \boldsymbol{M} = 1, \quad \forall \tilde{\boldsymbol{\Sigma}} \tag{73}$$

are naturally enforced. The same holds for the evolution equations corresponding to hardening. Obviously, additional constraints such as $Q_{\mathrm{i}} \geq 0$ can be easily taken into account as well. Inserting Eqs. (66)–(69) into Eq. (64) leads to

$$I_{\mathrm{inc}} = I_{\mathrm{inc}}(\boldsymbol{X}), \quad \text{with} \quad \boldsymbol{X} = [\tilde{\boldsymbol{\Sigma}}, \tilde{\boldsymbol{Q}}_{\mathrm{k}}, Q_{\mathrm{i}}, a] \quad \Rightarrow \quad \dim[\boldsymbol{X}] = 20 \tag{74}$$

Finally, the unknowns $\boldsymbol{X}$ can be computed from the constrained optimization scheme

$$\boldsymbol{X} = \arg \inf_{\boldsymbol{X}, \phi \leq 0} I_{\mathrm{inc}}(\boldsymbol{X}). \tag{75}$$

For that purpose, by now standard algorithms can be applied, cf. [38]. Evidently, the choice of a suited method depends strongly on the possibly non-linear constraint $\phi = 0$ (plastic loading). In the following paragraph, attention is turned on a certain class of plasticity models. This class contains a large number of important constitutive laws.

In this paragraph, an efficient solution scheme for optimization problem (75) is developed. It is restricted to yield functions of the type

$$\phi = \Sigma_{\mathrm{eq}}(\boldsymbol{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) - Q_{\mathrm{i}}(\alpha_{\mathrm{i}}) - Q_0^{\mathrm{eq}} \tag{76}$$

with Σ_{eq} denoting a positively homogeneous function of degree one. As mentioned before, many constitutive laws such as von Mises, Drucker-Prager or Rankine type models fall into the range of Eq. (76). According to Remark 1, in this case, the cumbersome non-linear constraint $\phi = 0$ can be directly included in the dissipation resulting in $\mathcal{D} = \lambda \; Q_0^{\mathrm{eq}} \geq 0$. Furthermore, since for this class of models the evolution equations yield

$$\boldsymbol{H}_{\mathrm{k}} = -\boldsymbol{M}, \quad \text{and} \quad \dot{\alpha}_{\mathrm{i}} = -\lambda, \tag{77}$$

the non-linear constrained optimization problem (75) can be significantly simplified, i.e.,

$$\boldsymbol{X} = \arg \inf_{\boldsymbol{X}} I_{\mathrm{inc}}(\boldsymbol{X}), \quad \text{with} \quad I_{\mathrm{inc}} = \Psi_{n+1}(\boldsymbol{X}) - \Psi_n + Q_0^{\mathrm{eq}} \, \Delta\lambda \tag{78}$$

with the unknowns being

$$\boldsymbol{X} = [\tilde{\boldsymbol{\Sigma}}, a] \quad \Rightarrow \quad \dim[\boldsymbol{X}] = 10. \tag{79}$$

As a result, the complexity of the problem is reduced by a factor of 2. The unconstrained minimization problem (78) can be solved in a standard manner, e.g., by employing gradient-type schemes, cf. [39]. Applying the time integrations (40) and subsequently, using the derivatives

$$\frac{\partial \Psi^{\mathrm{e}}}{\partial \Delta\lambda} = - \left[(\boldsymbol{F}^{\mathrm{e}}_{\mathrm{trial}})^T \cdot \frac{\partial \Psi^{\mathrm{e}}}{\partial \boldsymbol{F}^{\mathrm{e}}} \right] : D\exp\left[- \Delta\lambda \, \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}}\right] : \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}} \tag{80}$$

$$\frac{\partial \Psi^{\mathrm{e}}}{\partial \tilde{\boldsymbol{\Sigma}}} = - \left[(\boldsymbol{F}^{\mathrm{e}}_{\mathrm{trial}})^T \cdot \frac{\partial \Psi^{\mathrm{e}}}{\partial \boldsymbol{F}^{\mathrm{e}}} \right] : D\exp\left[- \Delta\lambda \, \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}}\right] : \partial^2_{\boldsymbol{\Sigma}}\phi\Big|_{\tilde{\boldsymbol{\Sigma}}} \; \Delta\lambda \tag{81}$$

$$\begin{aligned} \frac{\partial \Psi^{\mathrm{p}}}{\partial \Delta\lambda} &= \frac{\partial \Psi^{\mathrm{p}}}{\partial \alpha_{\mathrm{i}}} \frac{\partial \alpha_{\mathrm{i}}}{\partial \Delta\lambda} + \frac{\partial \Psi^{\mathrm{p}}}{\partial \boldsymbol{\alpha}_{\mathrm{k}}} : \frac{\partial \boldsymbol{\alpha}_{\mathrm{k}}}{\partial \Delta\lambda} \\ &= \quad Q_{\mathrm{i}} \quad + \boldsymbol{Q}_{\mathrm{k}} : \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}} \, , \end{aligned} \tag{82}$$

$$\frac{\partial \Psi^{\mathrm{p}}}{\partial \tilde{\boldsymbol{\Sigma}}} = \frac{\partial \Psi^{\mathrm{p}}}{\partial \boldsymbol{\alpha}_{\mathrm{k}}} : \frac{\partial \boldsymbol{\alpha}_{\mathrm{k}}}{\partial \tilde{\boldsymbol{\Sigma}}} = \Delta\lambda \, \boldsymbol{Q}_{\mathrm{k}} : \partial^2_{\boldsymbol{\Sigma}}\phi\Big|_{\tilde{\boldsymbol{\Sigma}}} \tag{83}$$

the gradient of I_{inc} can be computed in a straightforward manner. In line with Eq. (66), the elastic part of the deformation gradient is computed by means of

$$\boldsymbol{F}^{\mathrm{e}}_{n+1} = \boldsymbol{F}^{\mathrm{e}}_{\mathrm{trial}} \cdot \exp\left[-a^2 \;\; \partial_{\boldsymbol{\Sigma}}\phi|_{\tilde{\boldsymbol{\Sigma}}}\right] \tag{84}$$

Clearly, in case of an exponential approximation of the evolution equations for $\boldsymbol{\alpha}_{\mathrm{k}}$, the gradient has to be modified accordingly. In Eqs. (80) and (81), the derivative of the exponential mapping

$$D\exp\left[\boldsymbol{A}\right] := \frac{\partial \exp\left[\boldsymbol{A}\right]}{\partial \boldsymbol{A}} \tag{85}$$

can be computed in a standard fashion, e.g. [40,41]. The examples presented in the next section have been computed by applying a globally convergent Newton-type iteration, cf. [39]. For that purpose, the second derivatives of I_{inc} are required. Although they result in relatively lengthy equations, they can be calculated in a straightforward manner. Therefore, the Hessian of I_{inc} is omitted.

Remark 2 *According to Eqs. (80) and (83), stability of I_{inc} with respect to the* *plastic multiplier $\Delta\lambda$ reads*

$$\begin{aligned} \left.\frac{\partial I}{\partial \Delta\lambda}\right|_{\Delta\lambda=0} &= -\left[\boldsymbol{\Sigma} : \partial_{\boldsymbol{\Sigma}}\phi + Q_{\mathrm{i}} + \boldsymbol{Q}_{\mathrm{k}} : \partial_{\boldsymbol{\Sigma}}\phi + Q_0^{\mathrm{eq}}\right]|_{\mathrm{trial}} \\ &= -\phi_{\mathrm{trial}} > 0 \end{aligned} \tag{86}$$

which coincides with the classical (discrete) unloading condition $\phi_{\mathrm{trial}} \leq 0$. It is noteworthy that this property is fulfilled for any consistent time integration. Furthermore, even if the equivalent stress Σ_{eq} is not a positively homogeneous function of degree one, the inequality

$$-\left[(\boldsymbol{\Sigma} - \boldsymbol{Q}_{\mathrm{k}}) : \partial_{\boldsymbol{\Sigma}}\phi\right]|_{\mathrm{trial}} \leq -\Sigma_{\mathrm{eq}} \tag{87}$$

holds (ϕ is convex) and thus,

$$\left.\frac{\partial I}{\partial \Delta\lambda}\right|_{\Delta\lambda=0} \leq -\phi_{\mathrm{trial}} \tag{88}$$

According to Ineq. (88), the variational constitutive update tends to overestimate the elastic limiting loading, i.e., plastic loading occurs later (compared to standard plasticity theory) and thus, $\phi_{\mathrm{trial}} > 0$ represents a necessary loading criterion for the variational update. However, it has to be stressed once again that for Σ_{eq} being a positively homogeneous function of degree one, both schemes (conventional computational plasticity and the variationally consistent method as developed in the present paper) lead to identical loading conditions.

Remark 3 *Using a parameterization of the type $\Delta\lambda = a^2 \geq 0$, the functional I_{inc} shows an extremum at $a = 0$ (if $Q_{\mathrm{i}}(t_n) = 0$ and $\boldsymbol{Q}_{\mathrm{k}}(t_n) = \mathbf{0}$). Hence, if classical gradient-type optimization schemes are employed, a non-vanishing initial value $a = TOL > 0$ should be used for a plastic loading step.*

3.3.2 Fully isotropic models

In this subsection, a tuned version of the optimization scheme discussed before is given. It is based on the same assumptions as made before (Eq. (76)). Additionally, the stored energy potential Ψ as well as the yield function are postulated to be isotropic tensor functions (without structural tensors). For such models, it is straightforward to show that all tensors are coaxial. More precisely,

$$
\begin{aligned}
\boldsymbol{C}^{\mathrm{e}}_{\mathrm{trial}} &= \sum_{i=1}^{3} \left(\lambda_i^{\mathrm{etrial}}\right)^2 \; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \\
\boldsymbol{C}^{\mathrm{e}} &= \sum_{i=1}^{3} (\lambda_i)^2 \; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \\
\boldsymbol{\Sigma} &= \sum_{i=1}^{3} \Sigma_i \; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \\
\partial_{\boldsymbol{\Sigma}}\phi &= \sum_{i=1}^{3} \partial_{\Sigma_i}\phi \; \boldsymbol{N}_i \otimes \boldsymbol{N}_i
\end{aligned}
\tag{89}
$$

Consequently, the optimization scheme (78) reduces to

$$
\boldsymbol{X} = \arg\inf_{\boldsymbol{X}} I_{\mathrm{inc}}(\boldsymbol{X}), \quad \text{with} \quad I_{\mathrm{inc}} = \Psi_{n+1}(\boldsymbol{X}) - \Psi_n + Q_0^{\mathrm{eq}}\, a^2 \tag{90}
$$

with the unknowns being

$$
\boldsymbol{X} = [\tilde{\Sigma}_1, \tilde{\Sigma}_2, \tilde{\Sigma}_3, a] \quad \Rightarrow \quad \dim[\boldsymbol{X}] = 4. \tag{91}
$$

Again, it is strictly distinguished between principal Mandel stresses Σ_i and their pseudo counterparts $\tilde{\Sigma}_i$ which define the flow direction. Using the spectral decomposition of the total differential of the exponential mapping (for fixed eigenvectors $\boldsymbol{N}_i$)

$$
\begin{aligned}
\mathrm{d}\left\{\exp\left[-\Delta\lambda\; \partial_{\boldsymbol{\Sigma}}\phi\right]\right\} = &- \left\{\sum_{i=1}^{3} \partial_{\Sigma_i}\phi \; \exp[-\Delta\lambda\; \partial_{\Sigma_i}\phi]\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i\right\} \mathrm{d}\Delta\lambda \\
&- \sum_{j=1}^{3} \left\{\sum_{i=1}^{3} \Delta\lambda\; \partial^2_{\Sigma_i\Sigma_j}\phi \; \exp[-\Delta\lambda\; \partial_{\Sigma_i}\phi]\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i\right\} \mathrm{d}\Sigma_j,
\end{aligned}
\tag{92}
$$

together with

$$
\left[(\boldsymbol{F}^{\mathrm{e}}_{\mathrm{trial}})^T \cdot \frac{\partial\Psi^{\mathrm{e}}}{\partial\boldsymbol{F}^{\mathrm{e}}}\right] = \sum_{i=1}^{3} P_i^{\mathrm{e}}\; \lambda_i^{\mathrm{etrial}}\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \tag{93}
$$

the gradients (80) – (83) simplify greatly, i.e.,

$$
\frac{\partial\Psi^{\mathrm{e}}}{\partial\Delta\lambda} = -\sum_{i=1}^{3} P_i^{\mathrm{e}}\; \lambda_i^{\mathrm{etrial}}\; \partial_{\Sigma_i}\phi|_{\tilde{\Sigma}_i}\; \exp(-\Delta\lambda\; \partial_{\Sigma_i}\phi|_{\tilde{\Sigma}_i}) \tag{94}
$$

$$
\frac{\partial\Psi^{\mathrm{e}}}{\partial\tilde{\Sigma}_j} = -\sum_{i=1}^{3} P_i^{\mathrm{e}}\; \lambda_i^{\mathrm{etrial}}\; \Delta\lambda\; \partial^2_{\Sigma_i\Sigma_j}\phi\Big|_{\tilde{\Sigma}_i}\; \exp[-\Delta\lambda\; \partial_{\Sigma_i}\phi|_{\tilde{\Sigma}_i}] \tag{95}
$$

$$\frac{\partial \Psi^{\mathrm{p}}}{\partial \Delta\lambda} = Q_{\mathrm{i}} + \boldsymbol{Q}_{\mathrm{k}} : \left(\sum_{i=1}^{3} \partial_{\Sigma_i}\phi|_{\tilde{\Sigma}_i} \;\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \right) \tag{96}$$

$$\frac{\partial \Psi^{\mathrm{p}}}{\partial \tilde{\Sigma}_j} = \Delta\lambda \, \boldsymbol{Q}_{\mathrm{k}} : \left(\sum_{i=1}^{3} \partial_{\Sigma_i \Sigma_j}\phi^2\Big|_{\tilde{\Sigma}_i} \;\; \boldsymbol{N}_i \otimes \boldsymbol{N}_i \right) \tag{97}$$

Here, P_i^{e} represents the eigenvalues of $\boldsymbol{P}^{\mathrm{e}} := \partial_{\boldsymbol{F}^{\mathrm{e}}}\Psi$. If a Newton-type iteration is to be applied for solving the nonlinear optimization scheme, the second derivatives of I_{inc} are needed. As in the more general case discussed in the previous subsection, they can be computed in a straightforward manner. Consequently, they are omitted here. Clearly, analogously to the standard return-mapping algorithm formulated in principal axes, the now non-vanishing derivatives of the eigenvectors $\boldsymbol{N}_i$ have to be considered, i.e., $\mathrm{d}(\boldsymbol{N}_i \otimes \boldsymbol{N}_i) \neq \boldsymbol{0}$, cf. [1].

4 Numerical examples

The versatility and the performance of the proposed constitutive update are demonstrated by means of selected numerical examples. For the sake of concreteness, the prototype model as summarized in Subsection 2.3 is considered. Accordingly, isotropic as well as (linear) kinematic hardening are taken into account.

For the elastic response, a quadratic, orthotropic model characterized by the elastic stored energy potential

$$\begin{aligned} \Psi^{\mathrm{e}} = \quad & \frac{1}{2}\,\lambda\, J_1^2 + \mu\, J_2 + \frac{1}{2}\,\alpha_1\, J_4^2 + \frac{1}{2}\,\alpha_2\, J_6^2 + 2\,\alpha_3\, J_5 + 2\,\alpha_4\, J_7 \\ & + \alpha_5\, J_4\, J_1 + \alpha_6\, J_6\, J_1 + \alpha_7\, J_4\, J_6 \end{aligned} \tag{98}$$

is adopted, cf. [42]. Here, J_i are the invariants

$$\begin{gathered} J_1 := \mathrm{tr}[\boldsymbol{E}], \quad J_2 := \mathrm{tr}[\boldsymbol{E}^2], \\ J_4 := \mathrm{tr}[\boldsymbol{M}^{(1)} \cdot \boldsymbol{E}], \quad J_5 := \mathrm{tr}[\boldsymbol{M}^{(1)} \cdot \boldsymbol{E}^2], \quad J_6 := \mathrm{tr}[\boldsymbol{M}^{(2)} \cdot \boldsymbol{E}], \quad J_7 := \mathrm{tr}[\boldsymbol{M}^{(2)} \cdot \boldsymbol{E}^2] \end{gathered} \tag{99}$$

depending on the Green-Lagrange strain tensor $\boldsymbol{E}$ and so-called structural tensors $\boldsymbol{M}^{(i)} = \boldsymbol{m}_i \otimes \boldsymbol{m}_i$ where $\boldsymbol{m}_i$ span an orthonormal basis. In this section, the bases $\boldsymbol{m}_i$ are assumed to be of the type $\boldsymbol{m}_1 = [\cos\beta; \sin\beta; 0]$, $\boldsymbol{m}_2 = [-\sin\beta; \cos\beta; 0]$ and $\boldsymbol{m}_3 = [0; 0; 1]$. The angle β is set to $\beta = 10°$. The material parameters defining the elastic response are summarized in Tab. 1.

Following Subsection 2.3, the elastic space is defined by a Hill-type yield function, cf. (30). Thus, different material symmetries can be incorporated by choosing the

	λ	μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7
ortho.	67.25	81.00	67.46	-3.10	-15.00	0.00	2.00	-7.55	0.98
iso.	67.25	81.00	0	0	0	0	0	0	0

Table 1
Material parameters (GPa) defining the orthotropic and the isotropic elastic response according to Eq. (98), cf. [42]

4th-order weighting tensor $\mathbb{D}$ accordingly. In the numerical examples presented in this section, a fully isotropic and an orthotropic yield function are considered. The corresponding non-vanishing components of the weighting tensor $\mathbb{D}$ are given in Tab. 2. It can be easily checked, that the coefficients $\mathbb{D}_{ijkl}$ define indeed a convex

	$\mathbb{D}_{1111}$	$\mathbb{D}_{2121}$	$\mathbb{D}_{3131}$	$\mathbb{D}_{1212}$	$\mathbb{D}_{2222}$	$\mathbb{D}_{3232}$	$\mathbb{D}_{1313}$	$\mathbb{D}_{2323}$	$\mathbb{D}_{3333}$
ortho.	0.918887	4.18388	6.25	4.18388	-0.516313	5.0625	6.25	5.0625	5.84076
iso.	1	1	1	1	1	1	1	1	1

Table 2
Non-vanishing components of the weighting tensor $\mathbb{D}$ for an orthotropic as well as for an isotropic equivalent stress Σ^{eq} according to Eq. (33)

space of admissible elastic stresses. Different hardening models are analyzed. Each of them falls into the class defined by Eqs. (27) and (29). The respective material parameters are listed below.

	H_i	H_k
No hardening	0	0
Isotropic	1.0	0
Kinematic	0.0	1.0
Combined	0.5	0.5

Table 3
Material parameters (GPa) for different hardening models according to Eqs. (27) and (29)

4.1 *Shear test*

At first, a simple shear test is investigated. More precisely, the following stress states are analyzed:

$$\boldsymbol{P} = P_{12}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2, \quad \boldsymbol{P} = P_{13}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_3, \quad \boldsymbol{P} = P_{23}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_3. \tag{100}$$

Here and henceforth, $\boldsymbol{e}_i$ denote the standard cartesian basis. It bears emphasis that the vectors $\boldsymbol{e}_i$ are not identical to those defining the material symmetry, i.e., $\boldsymbol{e}_i \neq$

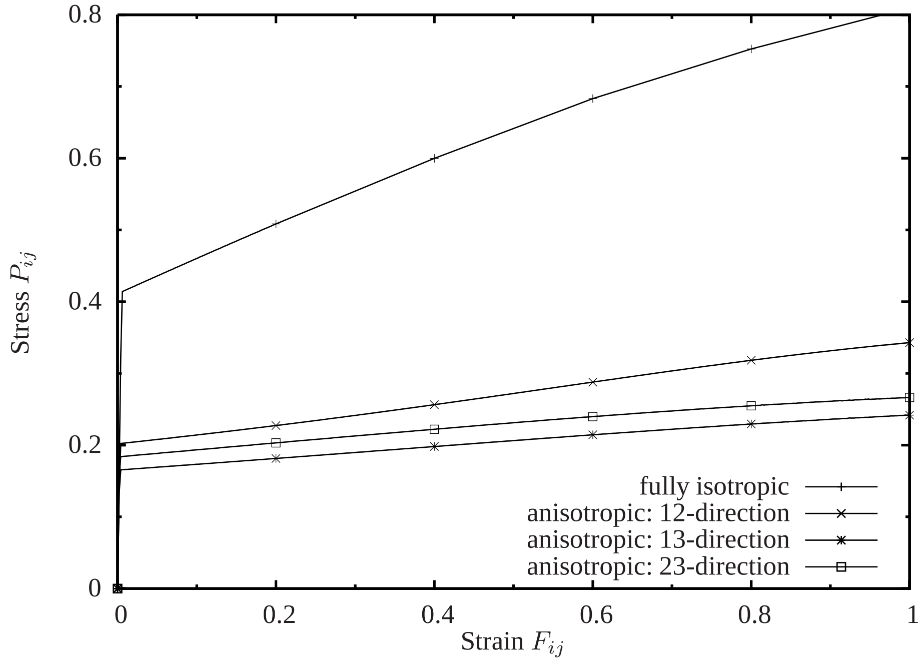

Fig. 1. Monotonic shear test: stress-strain diagram associated with isotropic hardening ($H_i = 1.0$, $H_k = 0.0$); results predicted by the fully anisotropic model (elastic and plastic) depending on the loading direction. For the sake of comparison, the response computed from the isotropic model (elastic and plastic) is shown as well.

$\boldsymbol{m}_i$. In contrast to purely displacement-driven problems, Eqs. (100) represent a Neumann problem. For the computation of the solution, a Newton-type iteration has been implemented. By doing so, the linearizations of the algorithm can be checked.

For a careful analysis of the fully orthotropic model (orthotropic elastic and plastic response), the three different simple shear tests according to Eqs. (100) are compared to the fully isotropic model. The computed results are shown in Fig. 1. Here, isotropic hardening has been assumed, cf. Tab. 3. It is evident from Fig. 1 that the model is indeed highly anisotropic. More precisely, depending on the loading direction, elastic yielding starts at different stress states. Furthermore, since the investigated problem is highly coupled, hardening is affected by the loading direction as well. However, this dependency is less pronounced.

For kinematic hardening (see Tab. 3), the stress-strain-diagrams corresponding to the simple shear test are shown in Fig. 2. Again, the orthotropic material behavior is obvious. However, in contrast to isotropic hardening, kinematical hardening induces an additional degree of coupling. For this reason, the differences in the $P - F$ diagrams are more pronounced compared to Fig. 1.

Next, the influence of different hardening models is investigated. For that purpose, cycling loading is considered. The computed responses for the fully isotropic model (elastically isotropic and isotropic yield function) are summarized in Fig. 3. As expected, during the first stage (loading) all hardening approaches lead to the same

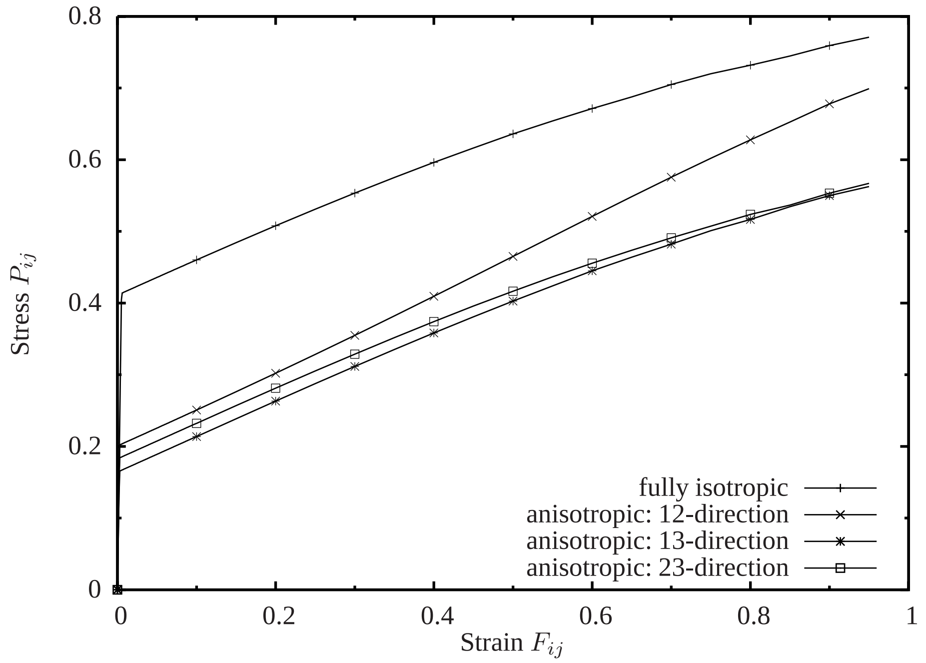

Fig. 2. Monotonic shear test: stress-strain diagram associated with kinematic hardening ($H_i = 0.0$, $H_k = 1.0$); results predicted by the fully anisotropic model (elastic and plastic) depending on the loading direction. For the sake of comparison, the response computed from the isotropic model (elastic and plastic) is shown as well.

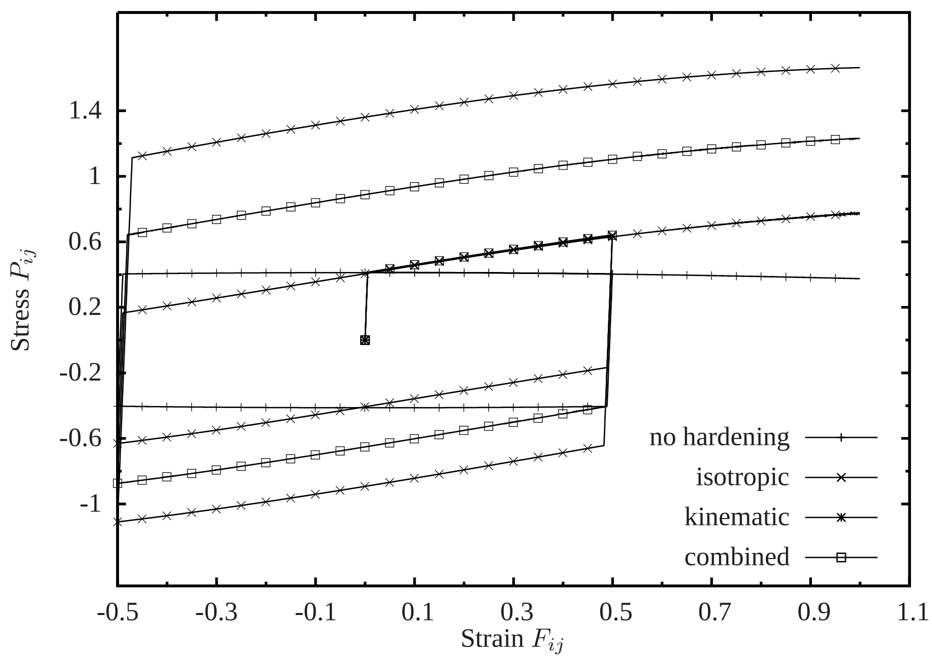

Fig. 3. Cyclic shear test: stress-strain diagram associated with a fully isotropic formulation (elastic and plastic) depending on the hardening model

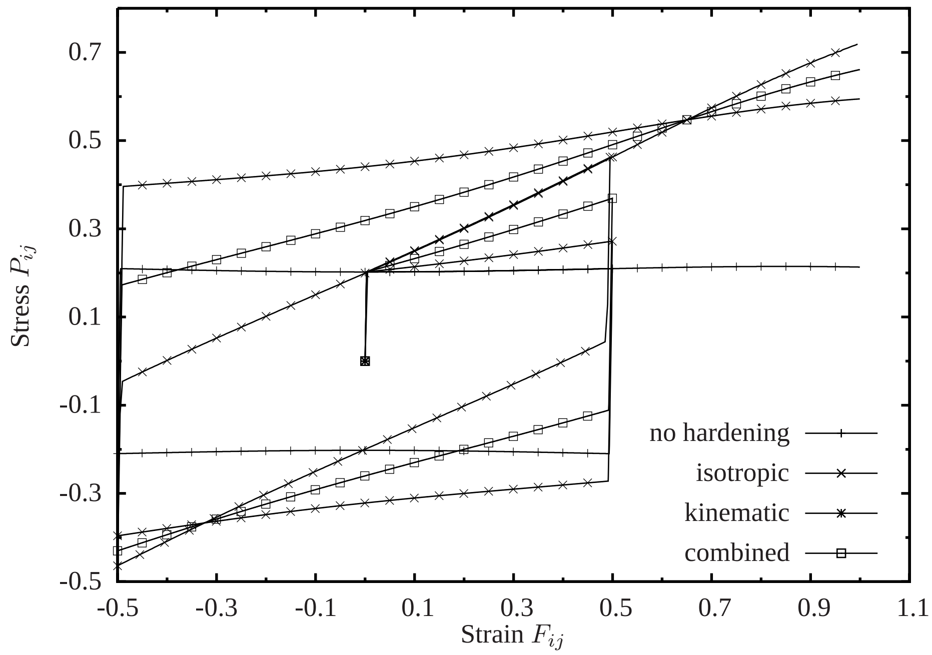

Fig. 4. Cyclic shear test: stress-strain diagram associated with a fully anisotropic formulation (elastic and plastic) depending on the hardening model

results, since the harding moduli are identical, cf. Tab. 3. Furthermore, after unloading and an additional re-loading step, the classical hysteresis can be observed. Although the applied strains are relatively large, Fig. 3 agrees reasonably with the linearized theory. In summary, it can be verified that the proposed variational constitutive update works correctly and efficiently.

Next, the cyclic simple shear test is re-analyzed by adopting the fully anisotropic constitutive model (elastically orthotropic and orthotropic yield function). The computed stress-strain responses are given in Fig. 4. According to Fig. 4 and in contrast to the fully isotropic model (see Fig. 3), the $P - F$ diagrams are now even different during the first loading stage. Clearly, this is a direct consequence, of the anisotropy of the material. Although the simple shear test represents one of the simplest mechanical problems, it is relatively difficult to estimate the influence of the material anisotropy. Therefore, the need for efficient numerical algorithms such as that discussed in the present paper is of utmost importance.

4.2 Uniaxial tension test

The second investigated example is the uniaxial tension test characterized by the stress tensors

$$\boldsymbol{P} = P_{11}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \quad \boldsymbol{P} = P_{22}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \quad \boldsymbol{P} = P_{33}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3. \tag{101}$$

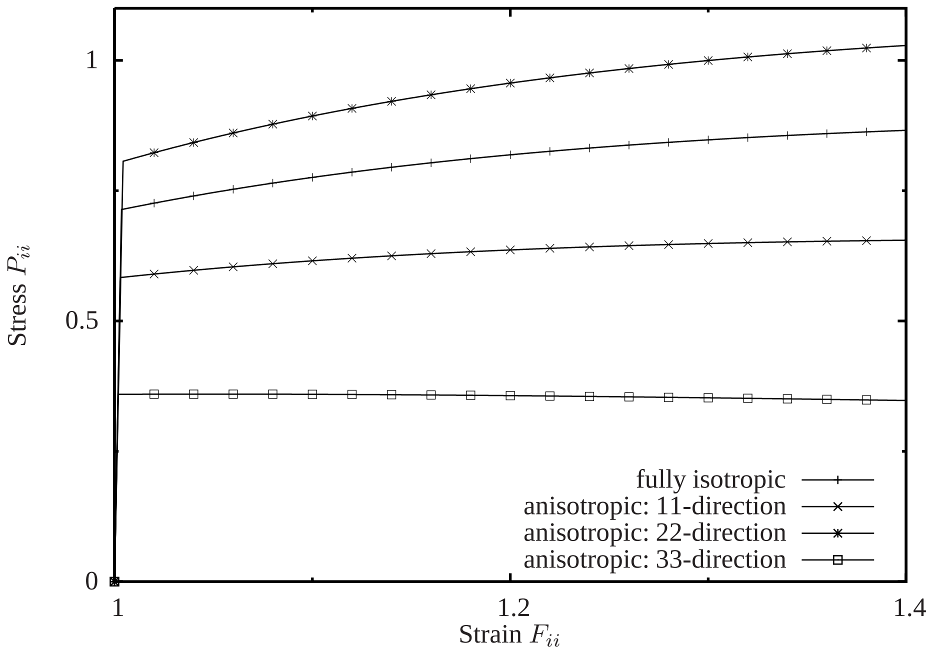

Fig. 5. Monotonic uniaxial tension test: stress-strain diagram associated with isotropic hardening ($H_i = 1.0$, $H_k = 0.0$); results predicted by the fully anisotropic model (elastic and plastic) depending on the loading direction. For the sake of comparison, the response computed from the isotropic model (elastic and plastic) is shown as well.

Again, the non-linear Neumann problem is solved by a Newton-type iteration.

In line with the previous subsection, monotonic tests are analyzed first. The computed results are summarized in Figs. 5 and 6. As for the shear test, the anisotropy of the elastic domain, together with the loading direction depending hardening, is evident.

The numerically computed response corresponding to cyclic loading is shown in Figs. 7 and 8. In analogy to the simple shear test, all hardening models lead to identical results during the first loading stage, if a fully isotropic model is considered. By contrast, according to Fig. 8, orthotropy induces an additional coupling through which the influence of hardening becomes highly-nonlinear and complex.

It is noteworthy that the proposed constitutive update improves significantly the robustness as well as the performance compared to conventional update schemes. For instance, the computation of initial values for a Newton-iteration is far from being straightforward. Such problems do not occur within the advocated method. Furthermore, even if the considered mechanical model is highly non-linear or non-smooth reliable and powerful optimization methods are available.

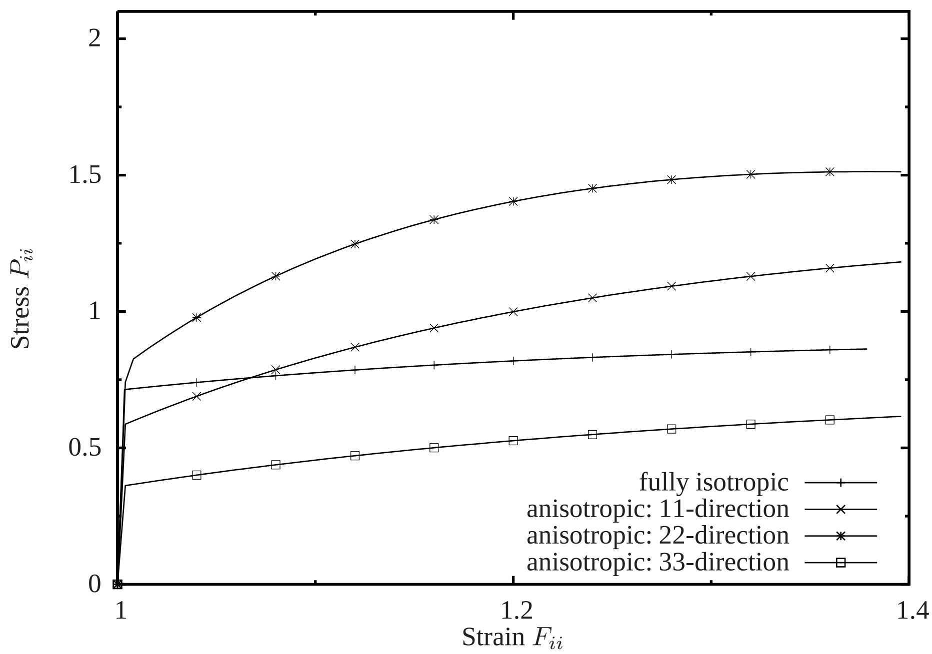

Fig. 6. Monotonic uniaxial tension test: stress-strain diagram associated with kinematic hardening ($H_i = 1.0$, $H_k = 0.0$); results predicted by the fully anisotropic model (elastic and plastic) depending on the loading direction. For the sake of comparison, the response computed from the isotropic model (elastic and plastic) is shown as well.

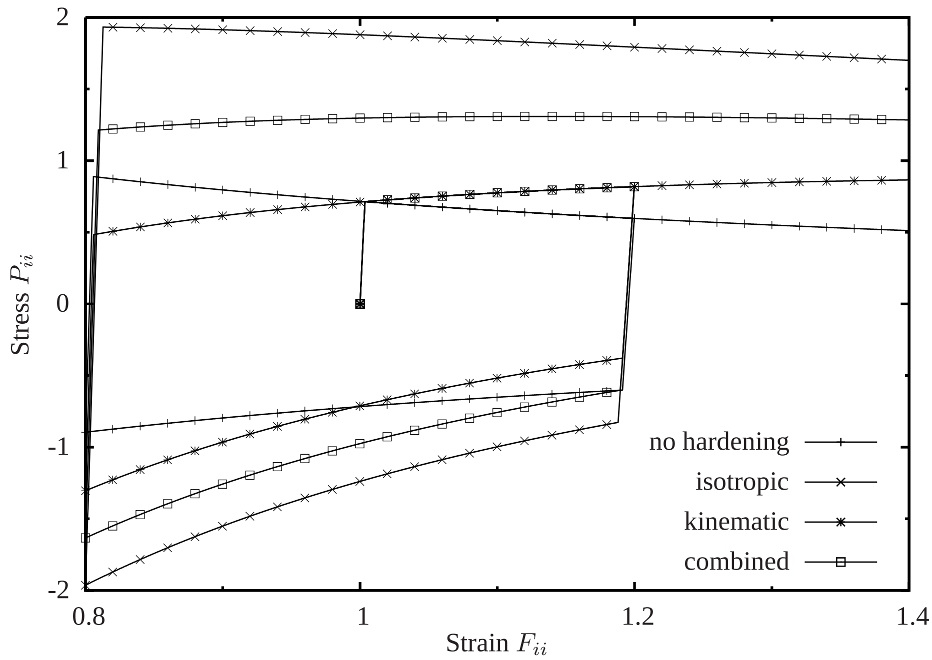

Fig. 7. Cyclic uniaxial tension test: stress-strain diagram associated with a fully isotropic formulation (elastic and plastic) depending on the hardening model

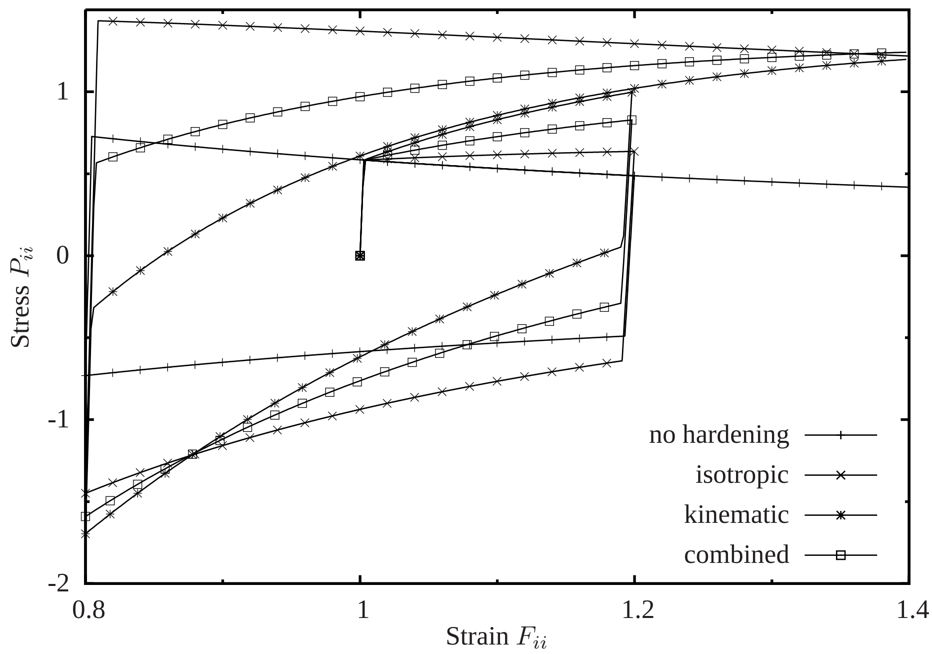

Fig. 8. Cyclic uniaxial tension test: stress-strain diagram associated with a fully anisotropic formulation (elastic and plastic) depending on the hardening model

5 Conclusions

An enhanced constitutive update for so-called standard dissipative solids has been proposed. In contrast to conventional update schemes such as the by now classical return-mapping algorithm, the new method is fully variational. More precisely, and in line with the previous works [13,16], the unknown history variables, together with the deformation mapping, follow jointly from minimizing an incrementally defined (energy) potential. Besides the associated mathematical and physical elegance, this method has some practical advantages. For instance, it allows to employ classical optimization methods for computing the solution of the aforementioned minimization problem. Unlike the prototype models advocated in [13,16], the proposed method covers a broad range of different constitutive models including anisotropic elasticity, anisotropic yield functions and isotropic as well as kinematic hardening. As a relatively complex example, an orthotropic Hill-type model including combined isotropic-kinematic hardening has been analyzed. It has been shown that although the investigated prototype represents a highly non-linear, coupled and anisotropic problem, the advocated variational constitutive update works very robustly and efficiently.

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